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Determination of the Temperature of Steam arising from Boiling Salt Solutions.

by

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Introduction.

Although it is well known that the boiling point of a salt solution is always higher than that of water under the same pressure, the temperature of steam arising from such a solution has been the subject of much dispute, and, even at present, different opinions seem to prevail among chemists and physicists.

According to one set of observers, the temperature of steam arising from a boiling salt solution is the same or nearly the same as that of the solution: Gay-Lussac and Faraday, and, subsequently, Wüllner, Magnus, and Pfaundler, rank among these. According to the other, it is the same or nearly the same as that coming from pure water boiling under the same pressure, whatever may be the temperature of the boiling solution: this is the view held by Rudberg and, to a certain extent, by Müller. Experimental proofs are adduced on both sides, but these, as I shall show, are none of them conclusive, and establish, I think, neither the one view nor the other.

The determination of the real temperature of steam arising from a boiling salt solution presents many experimental difficulties; for, besides others to be mentioned later on, steam condenses upon the bulb of the thermometer and also upon the sides of the vessel above

the vapour, unless special precautions be taken, so that the temperature indicated will not be higher than 100°. If, on the other hand, the walls of the containing vessel be maintained at a higher temperature, the thermometer will be exposed to the heat radiated from them, and will indicate a higher temperature.

For some time past, I have been engaged in an experimental investigation of the determination of the temperature of steam escaping from boiling salt solutions, and have devised a method which is, as far as I can see, free from such objections. By this means, it can be readily shown that the temperature of the steam escaping from the boiling salt solution is exactly the same as that of the solution itself.

Before proceeding, however, to a description of my own experiments, I shall briefly refer to the history of this question, discenssing the methods employed and the results obtained by previous workers.

Historical Summary and Short Critical Review of the Work already done.

Faraday, as long ago as 1822, published the results of his experiments upon this question in the *Annales de Chimie et de Physique* for that year. He found that when the bulb of a thermometer was sprinkled over with a salt and then introduced into steam coming out of boiling water, the thermometer showed a temperature higher than 100°, especially when it was held horizontally, so as to prevent the salt from being washed away too quickly. Still higher temperatures were observed by him when, the experiment being otherwise the same, the thermometer bulb was wrapped up in a linen or woollen cloth. From these experiments, Faraday concluded that since a salt solution was heated up to its boiling point by the action of steam at 100° upon the salt, therefore the steam generated from a boiling salt solution had only the temperature of 100°.

Gay-Lussac, as the editor of the French journal, appended a note to this paper, in which he first pointed out that facts similar to those observed by Faraday had long been known in France, namely, that when steam from boiling pure water was passed into a cold concentrated solution of a salt, the latter could be heated up nearly to its boiling point. Then, with regard to Faraday's view that steam generated from a boiling salt solution has only the temperature of 100°, Gay-Lussac remarked : "Sans invoquer ici le secours de la théorie, nous pouvons affirmer, d'après le témoignage irrécusable de l'expérience, que la temperature de la vapeur fournie par un liquide quelconque, sons une pression quelconque, est exactement celle de la couche liquide immédiatement en contact avec la vapeur."

Faraday then undertook more researches on this question, and in the *Quarterly Journal of Science* for 1823, he published the results of his experiments, and stated that he had proved Gay-Lussac's assertions to be correct, but that he had been astonished at the difficulty of obtaining definite results. Only when he used a double-walled vessel, which contained the experimental solution both between the walls and also above them, only when he heated the thermometer previously to a temperature higher than that of the boiling solution, and only after repeated observations, had he been able to convince himself that no anomaly existed in this phenomenon. These experiments of Faraday will be criticised presently along with those of Magnus.

Rudberg (*Ann. Chem. Phys.* [*Pogg.*], **34**, 257), in 1835, published the results of a long series of observations on the temperature of steam evolved from different solutions, boiling under different pressures, and pointed out that it is always the same as that of steam arising from pure water boiling under the same pressure. Some of his numbers are quoted below.

Pressure in mm.	Temperature of steam. Water.	Salt solution.	Salt.
761·54	100·06°	100·07°	$Ca(NO_3)_2$
763·37	100·12	100·13	,,
769·64	100·35	100·34	,,
769·17	100·33	100·36	KNO_3
763·20	100·49	100·50	,,

The experiments of Rudberg are always regarded as conclusive evidence that the temperature of steam issuing from a boiling salt solution is only 100°; but as they were carried out in the ordinary way without proper precautions against the steam cooling before reaching the thermometer, we must regard his results as unsatisfactory, in presence of the fact that with such precautions the temperature proves to be higher.

Wüllner (*Ann. Chem. Phys.* [*Pogg.*] **110**, 387), from determinations of the tension of steam arising from boiling salt solutions, pointed out that such steam, having less tension than that evolved from pure water at the same temperature, must be non-saturated, and that theoretically, therefore, it could not have the temperature of 100°.

Magnus (*Ann. Chem. Phys.* [*Pogg.*] **112**, 408) is generally regarded as having experimentally proved that steam from a boiling salt solution has a temperature equal, or nearly equal, to that of the solution itself. It will, therefore, be interesting to know exactly the nature of his evidence, and I propose to refer to his paper somewhat in detail. In the first part, Magnus mentions that Rudorff, who was working in his laboratory, had devised two methods of ascertaining the real temperature of steam coming out of a salt solution, the results showing that the temperature of the steam is above 100°.

One of these methods consists in dipping the thermometer into a boiling salt solution and, when it has attained the temperature of the

latter, in holding it in the steam over the solution. This method of ascertaining the temperature of steam is, as Magnus himself allows, by no means convincing; for, as already observed by Faraday and others, when such a thermometer is held in the steam arising from boiling pure water, temperatures higher than 100° are always indicated.

The other method consists in previously heating the bulb of a thermometer to a temperature higher than that of the boiling solution, and then introducting it into the steam. It is said that, in this way, temperatures approaching the boiling point of the solution are indicated. I have. however, repeatedly tried this method, and have fully convinced myself that if sufficient care is taken to protect the bulb of the thermometer against the splashing up of the hot solution, the temperature rapidly falls to 100°.

After referring to these two methods proposed by Rudorff, and after giving a good historical summary of the subject, to which I am greatly indebted for this part of my paper, Magnus goes on to describe his own method and the results obtained by its means. His apparatus consists of a double-walled metallic vessel, filled to a convenient height, both within and without the inner walls, with the experimental solution. Steam, generated by heating the solution from below, rises in the inner as well as in the outer chamber, and escapes by means of a lateral tube, also metallic and provided with two holes, one of which opens into the inner chamber and the other into the outer chamber. The object of having double chambers is to keep the inner one surrounded by hot steam, so as to prevent loss of heat by radiation. The temperature of the steam is indicated by means of a thermometer held horizontally and inserted in the lateral tube, its bulb being placed in the centre of the inner chamber. The following are the results obtained by Magnus with a solution of calcium chloride.

Temperature		
Of the solution.	Of the steam.	Difference.
107·0°	105·25°	1·75°
107·5	105·5	2·00
108·0	105·8	2·2
109·2	106·5	2·7
110·0	107·0	3·0
111·0	107·6	3·4
112·0	108·1	3·9
113·0	108·8	4·2
114·0	110·0	4·0
115·0	110·9	4·1
116·0	111·2	4·8

Magnus confesses that these numbers have no absolute value; but, at the same time, he regards it as proved that the temperature of the steam is nearly the same as that of the solution, the difference being due to the unavoidable loss of heat by radiation. Such is the conclusion Magnus has drawn from his experimental results, and this seems to have been accepted by chemists and physicists as being convincing. Even those who had held the opposite view seem never to have criticised the method Magnus employed, or the results he obtained.

It seems to me, however, that the conclusion he draws from the results of the experiments is open to criticism. As the double-walled vessel, which he employed, was made entirely of metal, its walls must have been heated up by conduction nearly to the temperature of the boiling solution it contained, and the steam enclosed by such walls must have had nearly the temperature of the latter; that is, a temperature more or less approaching that of the solution, and that quite independently of its temperature when generated. His num-

bers in the second column of the table may have indicated, therefore, not so much the temperature of the natural steam as that of the surrounding walls and of the steam superheated by them. Strange to say, Magnus seems to have been aware of this element of uncertainty in his results, and yet to have disregarded it. He writes:—"Soviel ist indess durch diese Versuche bewiesen, dass die Dämpfe, welche aus kochenden Salzlösungen aufsteigen, eine höhere Temperatur als 100° haben, und eine um so höhere, je höher die Temperatur der kochenden Lösung ist. Dass sie aber dieselbe Temperatur wie diese Lösung haben, ist mir nicht gelungen nachzuweisen, und ich zweifle dass diess möglich sein wird. Denn wenn die Wände des Dampfraumes auf der Temperatur der kochenden Lösung erhalten werden, so ist man, wie schon oben bemerkt, nicht sicher, dass nicht die Erwärmung von diesen Wänden hervorgebracht ist, haben dagegen die Wände eine niedrigere Temperatur, so wird auch die Temperatur des Thermometers niedriger ausfallen."

What then, it may pertinently be asked, is the value of his experimental results? It seems to me that Magnus erred in having the walls of his vessel of a highly conducting material, so that the thermometer would indicate the temperature of these walls, derived as it was from the solution and even from the heat of the lamp, and then to regard the temparature observed as being that of the steam. In fact, it can be readily shown that by keeping the walls of a vessel at a temperature higher than 100°, say at 110°, the steam issuing from water itself indicates a temperature almost equal to that of the walls.

The results obtained by Magnus, which have been regarded as the most weighty and conclusive experimental proof of the view that the steam arising from a boiling salt solution has the same or nearly the same temperature as the latter, thus appear to me to be valueless. and the same remark applies to Faraday's later experiments already

referred to, in which he employed a double-walled vessel containing the experimental salt solution in both chambers.

Müller (*Berichte d. Deutsch. Chem. Ges.* **9**, 1629), apparently not acquainted with Faraday's earlier experiments, made observations similar to his, and came to the same conclusion as Faraday first did, namely, that the steam generated from a boiling salt solution has only the temperature of 100°. He regards this view as being confirmed by the following considerations.

1st. The solution begins to boil (that is, steam bubbles freely rise to the surface and there burst) below its proper boiling point, and the temperature then slowly rises. The steam formed at first at 100° acts on the solution in the same manner as that passed from without, and heats it up to the boiling point.

2nd. Higher temperatures are indicated by salt solutions when boiling gently than when boiling violently; in the latter case, more steam bubbles coming in contact with the bulb of the thermometer. Similarly, a rise of temperature is observed on removing the flame after violent ebullition.

Müller's account of the first effect of heating a salt solution is so remarkable that I feel it incumbent to adhere closely to his words. He says: "A solution of calcium chloride whose boiling point is 126°, for example, is already completely boiling at 110°, and the temperature then rises to 126° in about half a minute" (*loc. cit.*, 1631). I have repeatedly tried similar experiments, but have never been able to confirm the statement that the solution, whose proper boiling point is 126°, *is completely boiling* at 110°, though it is perfectly true that the solution *begins to boil* at about this temperature. I confess, therefore, that it is a relief to me to be able to dispense with any discussion of Müller's interpretation of these phenomena, for whatever their signification may be, the fact that the observed

phenomena of *incipient* ebullition of a salt solution differ in no respect from those of pure water, except that of higher temperatures, deprives them of all value for throwing light upon the question at issue. A similar remark may be made with regard to Müller's other observations, namely, those which relate to the difference of temperature in the boiling solution, according as it is freely evolving the steam or not. It is a well-known fact that a thermometer dipped into a boiling liquid, say water, gives very irregular indications, higher temperatures being always indicated when steam is less freely escaping. In short, the facts observed by Müller are by no means limited to salt solutions, and his argument that the steam at the moment of its formation has only the temperature of 100° cannot, therefore, be regarded as in any degree satisfactory.

Wüllner (*Berichte d. Deutsch. Chem. Ges.* **10**, 256) wrote a note on Müller's paper, pointing out that the fact that a solution of calcium chloride may be heated up to its boiling point by the action of ordinary steam had already been known to Gay-Lussac and observed by Faraday, who on more careful experimental enquiry changed his first opinion, and came to the conclusion that the steam escaping from a boiling salt solution must have the temperature of the latter. He also points out again that such steam is not saturated, and that, therefore, it could not have the temperature of 100°.

Pfaundler (*Berichte d. Deutsch. Chem. Ges.* **10**, 463) objects to Müller's conclusion on much the same grounds as Wüllner, and fully supports the views of the latter. Further on, Pfaundler attempts to explain the fact observed by Magnus that the temperature of steam arising from a boiling salt solution is not quite the same as that of the latter ; this explanation, which is not easy to accept, is also not needed, as the facts observed by Magnus, and upon which this explanation is based, are, as already pointed out, by no means convin-

cing. My own experiments prove, moreover, that there exists no difference between the temperature of the boiling solution and that of the steam issuing from it.

The above summary should give some idea of the vagueness and unsatisfactory character of the experimental evidence now existing as regards the temperature of the steam arising from a boiling salt solution. The following is a description of my own experimental methods, and the results obtained by their means.

Experimental Methods.

For success in establishing the true temperature of the steam escaping from boiling salt solutions, certain conditions must be observed.

1. The thermometer must be kept clear of all contact with the solution, even the smallest drops thrown up by ebullition, as otherwise it is evident that the experiment loses all claim to accuracy.

2. The effect of the radiant cooling of the thermometer must either be prevented, or rendered inappreciable in proportion to the heating up by the steam. Before considering these alternative conditions, it may be well to call attention to the familiar and striking evidence we have that loss of heat by radiation from the bulb of the thermometer does occur and cannot be neglected. Whenever distillation of water, or any other liquid with no fixed matters dissolved in it, is going on, the thermometer immersed in the vapour to record the "boiling point" is seen to be always condensing some of the vapour, drops falling from it into the boiling liquid.

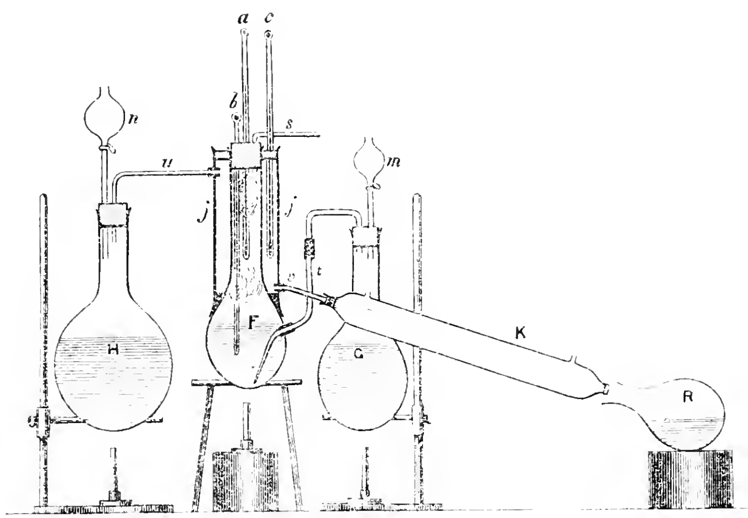
The former alternative seems hardly practicable, or, at least, has proved to be impracticable under various conditions in my hands. The latter alternative, that of overpowering the loss of heat by

radiation by rapid removal of the vapour in contact with the thermometer, can be easily effected by the expedient of combining the introduction of steam from without with the boiling of the solution by the lamp, the combination being regulated by maintaining steady the temperature marked by the thermometer in the solution. Ebullition alone should suffice for this purpose, but the practical difficulties in the way render it insufficient. The boiling would have to be tumultuous to generate much vapour, and in a short time the solution would become too concentrated for the experiment to be continued with convenience. On the other hand, with due regulation of the steam entering from without and the height of the flame, an abundant supply of steam can be got without impediment. Moreover, irregular boiling and bumping of the solution are both entirely prevented when the operation is worked in this manner.

3. The walls of that part of the vessel which serves as the steam chamber for the thermometer must be sufficiently protected from cooling externally, and yet, at the same time, not be heated to the temperature of the steam. For if, as in Rudberg's experiments, the former condition is not observed, so much of the steam is condensed in keeping the walls at 100° that it is hardly possible to keep enough passing over the thermometer bulb; while if the latter condition is ignored, as in Magnus's experiments, the indications of the thermometer may be illusory.

The arrangement which I adopted after several trials and modifications in order to meet these conditions was as follows. It consisted of an elongated, round-bottomed flask, F, with a long neck; this flask contained the experimental salt solution. The mouth of the flask was provided with a cork, through which passed two thermometers, a and b, and a tube s for the escape of steam; the thermometer a was used to indicate the temperature of the steam, and the thermometer

b that of the solution. The neck and a part of the flask F was en-



closed in a glass cylinder, *jj*, fixed above by means of a cork and below by means of an india-rubber band. Through the cork of the cylinder, and by the side of the neck of the flask F, passed a third thermometer, *c*, whose bulb was kept at the same height as that of the steam thermometer *a*. The body of the flask, finally, was provided with a hole in its side, through which a glass tube,* *t*, somewhat drawn out at one of its ends and bent as shown in the figure, was passed and jointed to the flask by means of a short piece of india-rubber tubing slipping over it. This tube, whose drawn-out end nearly reached the bottom of the flask, was used for supplying steam from the boiler G. The glass cylinder *jj* was connected on one hand with another flask, H, and on the other with a condenser, K; these connections were made as before by means of short pieces of india-rubber tubing slipped on over the glass tubes *u* and *v* respectively. The flask H contained acetic acid somewhat diluted with water.

The flasks F, G, and H were supported on stands, and F rested on a piece of asbestos cardboard, having a hole in the centre, so that only the bottom might be heated by the direct flame. The stoppered funnels *m* and *n* served to supply the flasks G and H with water and acetic acid respectively, when necessary, without dismounting the apparatus. The funnel *m* further served, at the end of an experiment, to restore the equilibrium of pressure between the outer air and the inside of the flask G, thus preventing the salt solution from being sucked into the latter. The whole arrangement is exceedingly simple, and can be fitted up by any one with materials found in every chemical laboratory. The three thermometers *a*, *b*, and *c* had, finally, been carefully verified, including the exposure correction for *a* and *c*.

The salt solution, whose boiling point had been approximately determined and known to be higher than that of the dilute acetic acid, was introduced into the flask F by means of a long funnel, and the inside of the neck of the flask then wiped with a cloth, so that no particle of the salt solution should be left adhering to it. Some cotton-wool was loosely tied round the stem of the thermometer *b*, and below the bulb of the thermometer *a*; this served as a very effective screen against splashing up of the solution on to the bulb of the second thermometer *a*. A little cotton wool was also tied round the stem of the thermometer *a*, and above its bulb; this prevented any water which might condense in the steam-issue tube *s*, from flowing down the stem of this thermometer. The cork with its two thermometers thus prepared was then fitted into the mouth of the flask F; this cork projected a little below that of the jacket, in order that there might be no chance of steam condensing within the neck of the flask. The tubes *u* and *t* were covered with cotton-wool in order to lessen the cooling action of the air on the acetic acid vapour and the steam respectively.

The acetic acid was now made to boil in the flask H. Its vapour, passing into the jacket *jj*, heated up the walls of the neck of the flask, that is, of the steam chamber over the solution, then passed into the condenser K, and collected in the receiver R. The flame under the flask H was so regulated that the acetic acid only slowly distilled ; in this manner, a steady supply of vapour of almost constant temperature could be readily maintained in the jacket. The thermometer *c* rapidly rose to about 111° under the influence of the acetic acid vapour, whilst the thermometer *a* more slowly rose, until ultimately it also showed about the same temperature.

The salt solution was next heated to boiling. The water in the flask G, which had been kept nearly boiling, was then also made to boil regularly, and a rapid current of steam was passed into the salt solution boiling in the flask F. The thermometer *a* now began to rise above that in the jacket, until it indicated the same temperature as that of the solution. The flame under the boiling solution was so regulated that the temperature of the latter should either remain constant or rise very slowly. The results obtained in one experiment with a solution of calcium chloride are shown below.

Temperature of the			Difference.	
Steam. (I.)	Solution. (II.)	Acetic acid vapour. (III.)	(II.)—(I.)	(I.)—(III.)
111·2°	112·5°	110·8°	1·3	0·4
111·7	112·5	110·9	0·8	0·8
112·2	112·6	111·1	0·4	1·1
112·5	112·7	111·3	0·2	1·2
112·7	112·9	111·5	0·2	1·2
113·0	113·0	111·6	0·0	1·4
113·1	113·2	111·8	0·1	1·3
113·3	113·3	111·9	0·0	1·4

The time occupied was about 20 minutes. The bulb of the steam thermometer *a* remained perfectly clean; indeed, by washing it with a little water and adding a drop of silver nitrate solution to the washings, not a trace of cloudiness was produced, showing that the splashing of the solution on to it had been completely prevented. In fact, the upper portion of the cotton-wool remained perfectly clean and dry.

For the following two experiments, still more dilute acetic acid was employed. The results obtained with a solution of sodium nitrate are as follows:—

Temperature of the			Difference.	
Steam. (I.)	Solution. (II.)	Acetic acid vapour. (III.)	(II.)—(I.)	(I.)—(III.)
106·6°	107·5°	105·9°	0·9	0·7
106·9	107·6	105·9	0·7	1·0
107·2	107·6	106·0	0·4	1·2
107·5	107·7	106·1	0·2	1·4
107·7	107·8	106·2	0·1	1·5
107·7	107·8	106·2	0·1	1·5
107·8	107·8	106·3	0·0	1·5

The time occupied was again about 20 minutes. The following are the observations made with a solution of potassium nitrate:—

Temperature of the Steam. (I.)	Solution. (II.)	Acetic acid vapour. (III.)	Difference. (II.)—(I.)	(I.)—(III.)
107·8°	109·4°	107·8°	1·6	0·0
108·3	109·5	107·8	1·2	0·5
108·7	109·7	108·0	1·0	0·7
109·1	109·8	108·1	0·7	1·0
109·3	109·8	108·2	0·5	1·1
109·5	109·9	108·2	0·4	1·3
109·8	110·0	108.2	0·2	1·6
110·0	110·2	108·3	0·2	1·7
110·2	110·3	108·5	0·1	1·8
110·2	110·3	108·5	0·1	1·7

The time occupied was about half an hour.

In the following experiment a stronger solution of calcium chloride was taken, and amyl alcohol (containing a little of the lower alcohols) was employed instead of acetic acid.

Temperature of the Steam. (I.)	Solution. (II.)	Amyl alcohol vapour. (III.)	Difference. (II.)—(I.)	(I.)—(III.)
127·5°	128·3°	126·5°	0·8	1·0
128·1	128·5	126·8	0·4	1·3
128·6	128·9	127·2	0·3	1·4
129·1	129·2	127·8	0·1	1·3
129·4	129·5	128·1	0·1	1·3
129·7	129·8	128·3	0·1	1·4
129·8	129·8	128·5	0·0	1·3

The time occupied was about 20 minutes.

The experiments above described prove, beyond any possible doubt, that *the temperature of the steam escaping from a boiling salt solution is*

exactly the same as that of the solution. This, I believe, is the first occasion on which the above important fact has been experimentally established. Philosophers had, indeed, asserted that the temperature of such steam should be, or would be, the same as that of the solution, but without any experimental proof. Gay-Lussac's assertion, "the temperature of the vapour furnished by any liquid is exactly the same as that of the liquid layer in immediate contact with the vapour," is based upon the knowledge of the facts derived from the study of simple liquids, such as water, alcohol, &c.; for no direct experiments had been made by him, or any other investigator before him, upon the temperature of the steam arising from a boiling *salt solution* which established the truth of that assertion. So far as direct experimental evidence is concerned, he had, I think, no good reason for extending his remarks to the case of a salt solution. Faraday thought he had proved Gay-Lussac's assertion to be correct in the case of salt solutions also, but upon evidence not at all convincing. Magnus could not show that the temperature of the steam is exactly the same as that of the solution. "That, however, it (the steam) possesses the same temperature as the solution I cannot prove, and I doubt whether it can possibly be proved," he says. The grounds for such doubt have now been removed by the experiments above described, the secret of the success lying in the fact that *the walls of the steam-chamber must be above* 100°, *and yet below the temperature of the solution, and that, at the same time, a sufficient quantity of steam must escape from the solution to ensure that these walls shall have no material cooling effect upon the steam.* To meet this condition the quantity of steam evolved from the boiling solution itself is not sufficient, and hence the necessity of passing steam from without into the boiling solution.

That the quantity of steam merely arising from a boiling salt

solution is not sufficient to overcome the cooling effect of the walls will best be seen from the following numbers. These refer to a solution of calcium chloride experimented upon in exactly the same manner as before, but without introduction of steam from an external source. The acetic acid was also more rapidly distilled.

Temperature of the			Difference.	
Steam. (I.)	Solution. (II.)	Acetic acid vapour. (III.)	(II.)—(I.)	(III.)—(I.)
103·0°	111·9°	103·2°	8·9	0·2
103·4	114·4	103·5	11·0	0·1
103·7	115·2	103·7	11·5	0·0
104·1	117·0	104·2	13·0	0·1
104·5	118·4	104·5	13·9	0·0
104·8	120·5	105·1	15·7	0·3
104·9	122·6	105·3	17·7	0·4
105·1	125·2	105·5	20·1	0·4

In spite of the fact that the solution was boiling briskly with a free and uninterrupted evolution of steam, and in spite of the additional fact that the temperature of the boiling solution rose from 111·9° to 125·2°, the temperature of the steam only rose from 103° to 105·1°, and never above that of the acetic acid vapour in the jacket. The rise that did occur is evidently due to the heating effect of the acetic acid vapour; for, in this case, the outer surface of the steam-chamber was constantly wet, owing to the condensation of the acetic acid vapour upon it. Acetic acid vapour, and not steam, was then acting as the heater.

This result was so remarkable and apparently so conclusive, that I was for some time induced to believe that the temperature of the steam escaping from a solution of calcium chloride boiling at the temperature of even 125° is lower than 105° and probably only 100°.

By slightly modifying the experiment, namely, by keeping the distillation of the acetic acid at a very slow rate, while the solution was kept boiling as briskly as possible, the steam-thermometer could be made to indicate somewhat higher temperatures than the acetic acid vapour, but far below the temperature of the solution. It was this observation, however, that led me to try the introduction of steam into the solution from without.

By the introduction of steam into the boiling solution from without, evaporation and condensation of steam in the solution can be so readily and exactly counterbalanced, that its boiling temperature may be maintained constant for any length of time and within a few thousandths of a degree Centigrade. I am developing this part of my observation for the exact determination of the boiling points of various solutions, and also for simplifying the determination of molecular weights by the boiling method, which has, of late, been made the subject of an extensive study by Beckmann.

In conclusion I wish to express my warmest thanks to my colleague Dr. E. Divers, F. R. S., for many valuable criticisms and suggestions from time to time while this investigation was in progress, and also to Dr. C. G. Knott and Professor K. Yamakawa for the interest which they have taken in my work.

Note on an observation by Gerlach of the Boiling Point of a solution of Glauber's salt.

by

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A few years ago Dr. G. T. Gerlach (*Zeit. anal. Chem.*, **26**, 413) published a paper in which he mentions that steam, escaping from a boiling solution of Glauber's Salt containing a crystalline magma of the anhydrous salt, shows the temperature of 100°, whilst the liquid is boiling at 82° or even at 72°. This observation appeared to me so curious and so anomalous that I was induced to repeat his experiments: the results, on the whole, confirmed Gerlach's observations as to temperatures but, at the same time, deprived them of all exceptional character. Gerlach describes his experiments in the following words:

"700 gr. of crystallized Glauber's salt were melted in an iron vessel and kept boiling for some time. The vessel was then removed from the lamp, and the liquid portion of its contents was poured off as completely as possible, whilst the separated anhydrous salt was left in the vessel. The whole was allowed to cool to about 50°, before the vessel was again heated. This vessel was now provided with a tin-plate cover having two holes, through one of which a thermometer passed nearly reaching the bottom of the vessel, and through the other a second thermometer which hang in the steam chamber over the heated crystalline magma."

"The liquid completely boiled at 82° (Die Flussigkeit kam schon bei 82°c. vollständig in's Kochen), whilst the escaping steam showed

the temperature of 100°. Only very slowly did the temperature of the heated magma rise and attained the temperature of about 100° as the mass got almost dry."

"To this almost dry mass I added 100 gr. of crystallized Glauber's salt, which melted on shaking."

"The crystalline magma now boiled at even 72°c., whilst the steam indicated 100°c." (*loc. cit.*, 422).

It is not correct to say that "the liquid completely boils at 82°, whilst the escaping steam shows the temperature of 100°", for it is only a wet mass of anhydrous sodium sulphate that is heated. I found that steam, in such a case, does not arise from the heated mass uniformly, but escapes through a number of channels produced in those portions of it which are in contact with the sides of the vessel and which are, therefore, most heated. The central portion of the magma, where the thermometer bulb finds itself, is more slowly heated, and hence the fact that while the steam indicates the temperature of 100°, the wet magma shows a lower temperature.

In the second experiment, where some crystallized Glauber's salt is added to the heated magma, the temperature of the latter is, for a time, very much lowered, not only because a quantity of a cold body is introduced, but also because the fusion of the crystallized salt absorbs much heat. The central portions of the mass, therefore, show as low a temperature as 72°, whilst steam which rapidly forms and escapes along the sides of the vessel shows the temperature of 100°. That the latter does not indicate a higher temperature is easy to understand from the form of the experiment. It is also needless to mention that the temperature of the whole mass soon rises.

Gerlach's observations are erroneous, then, in so far as they imply that a substance can evolve a vapour hotter than itself.

Modification of Beckmann's Boiling Method of determining Molecular Weights of Substances in Solution.

by

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With Plate I.

Recent investigations in Physical Chemistry have extended our means of determining molecular weights of substances. Besides the ordinary method, which depends upon the determination of the specific gravity of a substance in the gaseous state—the so-called "vapour density" method—we have now, thanks to the labours of Raoult and others, various means of ascertaining molecular weights of substances in solution. Diminution of vapour tension, lowering of the freezing point and rise of the boiling point, of a solvent, are the changes which accompany the dissolution of a solid body in it, and each of these changes supplies us with a means of determining the molecular weight of the dissolved body. These are mutually connected together by theoretical considerations, but for purposes of determining molecular weights the method, which depends upon the change in the boiling point, far surpasses the others on account of the simplicity of the manipulation and of the greater exactness in the results obtained. To Beckmann (*Zeitsch. physik. Chem.*, **3**, 603 [1889]; **4**, 532 [1880]; **5**, 76 [1890]; **6**, 437 [1890]; **8**, 223 [1891]) we owe the application of this method for laboratory purposes; but, in spite of his indefatigable labours, there are some points

in his method which are capable of improvement and simplification. The object of the present paper is to describe a modification of the method which renders it practicable in every laboratory and, at the same time, able to yield results which are even more concordant than those obtained by Beckmann.

It may, however, not be out of place to first briefly refer to the principle which underlies this method of determining molecular weights. By the study of the lowering of vapour pressure of various solvents by bodies dissolved in them, Raoult was ultimately led to the following law:

"The relative lowering of vapour pressure is proportional to the ratio of the number of molecules of the dissolved substance to the total number of molecules in the solution."

Expressed in a formula, we have

$$\frac{p-p'}{p} = c \times \frac{n}{N+n},$$

where p and p' are the vapour pressures of the solvent and of the solution, and N and n, the number of molecules of the solvent and of the dissolved body respectively. The constant c may be taken as equal to unity. If, then, G and g be the weights taken, and M and m the molecular weights, of the solvent and of the dissolved body, all expressed in grams, we have

$$\frac{p-p'}{p} = \frac{\frac{g}{m}}{\frac{G}{M} + \frac{g}{m}},$$

or

$$m = M.\frac{g}{G} \cdot \frac{p'}{p-p'}.$$

Hence, if M, G and g are known, the molecular weight of the dissolved substance, m, is given by determining p and p', the vapour pressures of the solvent and of the solution.

Now, the diminution of vapour pressure is proportional, on the

one hand, to the lowering of the freezing point and, on the other, to the rise of the boiling point. The former relation, which was established by Raoult in a purely empirical way (*Compt. rend.*, **87** [1878]; **103** [1886]; **104** [1887]; **105** [1887]; **107** [1888]; *Ann. Chim. Phys.* [6], **15** [1888]), had already been deduced by Guldberg (*Compt. rend.*, **70** [1870]) from the mechanical theory of heat. Moreover, van't Hoff (*Zeitsch. physik. Chem.*, **1** [1887]) theoretically deduced the laws of the lowering of the freezing points of solutions from the relation which exists between vapour pressure and osmotic pressure.

The determination of, the other relation, the rise of the boiling point also affords us a means of ascertaining the molecular weight of the dissolved substance, inasmuch as it is possible to determine the temperatures at which the solvent and the solution exhibit equal vapour pressures, instead of ascertaining their vapour pressures at an equal temperature. This method was first practically carried out by Beckmann.

The calculation of the molecular weight is made by taking into consideration the proportionality which exists between the difference of pressure and the difference of temperature. Now, since the solutions subjected to examination are all very dilute, the equation

$$\frac{p-p'}{p} = \frac{n}{N+n}$$

may assume the following form without causing any material difference in the results:

$$\frac{p-p'}{p} = \frac{n}{N}.$$

The difference of temperature produced in the boiling point by the dissolution of one gram-molecule of the substance in one gram molecule of the solvent and which corresponds to the above pressure-

difference is called the *molecular elevation of boiling point.* This constant is obtained experimentally by determining the rise of boiling point of a solvent, following the dissolution of a known weight of a substance of known molecular weight in a known quantity of the solvent. Thus, for example, Beckmann (*Zeitsch. physik. Chem.*, **3**, 603 [1889]) found that 4.740 grams of ethyl benzoate dissolved in 100 grams of ether raised the boiling point of the latter by 0.665°. The weight of ethyl benzoate dissolved in one gram-molecule of ether (74 grams) is, therefore, $\frac{4.740 \times 74}{100} = 3.5076$ grams. The rise of boiling point corresponding to the dissolution of one gram-molecule of ethyl benzoate (150 grams) in one gram-molecule of ether would then be

$$\frac{0.665 \times 150}{3.5076} = 28.4.$$

Experiments with other substances give nearly the same value for the molecular elevation of the boiling point of ether. The molecular weight of any dissolved substance is, then, obtained from the equation,

$$m = B \times \frac{g}{\Delta},$$

where, in case of ether, $B = 28.4$, $g =$ weight of the substance dissolved in 74 grams of ether, and $\Delta =$ the observed rise of boiling point.

In practice, it is more convenient to refer the constant B to 100 grams, instead of to the molecular weight, of the solvent; the calculation is thus very much simplified. The formula then becomes

$$m = B' \times \frac{g'}{\Delta},$$

where, in case of ether, $B' = 28.4 \times \frac{74}{100} = 21.0$, and $g' =$ weight of the substance dissolved in 100 grams of the solvent. As the mean of several determinations, Beckmann (*loc. cit.*) obtained the number, 20.9 for the value of B.

The constant B or B' may, moreover, be accurately calculated in the following manner from the latent heat of vaporization and the boiling point of the solvent, as has been pointed out by Arrhenius (*Zeitsch. physik. Chem.*, **4**, 550 [1889]).

Suppose we have a solvent of molecular weight, M, and of boiling point, T (in absolute temperature), at pressure, p. By dissolution of n gram-molecules in 100 grams of the solvent, the boiling point rises to $T+dT$. The relation between this rise, dT, and the dissolved mass, n, may be calculated thus:

The solution boils under pressure p at the temperature $T+dT$. The vapour pressure of the solvent at T is equal to p, it is equal to $p+dp$ at the temperature $T+dT$. Between dT and dp the following relation exists according to the second law of thermodynamics, as already shown by van't Hoff (*Zeitsch. physik. Chem.*, **1**, 494 [1887]):

$$\frac{dp}{p}=dT.\frac{\omega}{2T^2},$$

where $\omega=$ latent heat of vaporization of a gram-molecule of the solvent at temperature, T.

Now, according to Raoult's law, the relation between the diminution of vapour pressure and the number of molecules of the solvent and of the dissolved substance is expressed by

$$\frac{p-p'}{p}=\frac{n}{N+n},$$

or

$$\frac{(p+dp)-p}{p+dp}=\frac{n}{N+n},$$

or

$$\frac{dp}{p+dp}=\frac{n}{N+n}.$$

But, as the solution is very dilute, dp is very small compared to p, and n very small compared to N. The last equation may, therefore, be written as

$$\frac{dp}{p}=\frac{n}{N}$$

or, as 100 grams of the solvent is employed, and M is its molecular weight, we have

$$\frac{dp}{p}=n.\ \frac{M}{100}$$

Equating this value of $\frac{dp}{p}$ with that already obtained, we have

$$\frac{n.\ M}{100}=\frac{\omega.\ dT}{2T^2}.$$

In this equation, ω being the latent heat of vaporization of a gram-molecule of the solvent (M grams) in gram-calori, it follows that $\frac{\omega}{M}=$ latent heat of vaporization of one gram of the solvent. Calling this W, we have

$$\frac{n.\ M}{100}=\frac{M.\ W.\ dT}{2T^2}$$

or

$$dT=n\times\frac{.02T^2}{W}.$$

If $s=$ weight of the substance dissolved in 100 grams of the solvent and $m=$ molecular weight of the dissolved substance, then $n=\frac{s}{m}\cdot$ Substituting this value of n in the above equation and solving for m, we have

$$m=\frac{.02T^2}{W}.\ \frac{s}{dT}.$$

In this equation, s and dT have the same meaning as g' and Δ respectively in one of the preceeding equations, and the constant, $\frac{.02T^2}{W}$, for ether is found to be equal to

$$\frac{.02\times(273+34.97)^2}{90.1}=21.0,$$

the value which we already obtained for B' in the same equation.

Now, turning to the practical side of the question, the great and

the only difficulty which lies on the way is the exact determination of the boiling points, as has already been pointed out by Beckmann. Irregular boiling and bumping, attended with sudden alterations in the temperature of a boiling liquid are phenomena which are too well known to be mentioned. Even when the liquid seems to be regularly boiling, there are still constant changes in the temperature of the liquid, as may be readily observed by the use of a delicate thermometer. Various means have been tried to overcome these difficulties. Bits of metallic wire or foil, of broken glass, etc. are almost useless. Raoult tried coating the thermometer bulb with a layer of palladium recently charged with hydrogen gas, but it did not lead to the desired result, as the evolution of hydrogen gas soon slackened and, in about 20 minutes, stopped altogether.

Beckmann has succeeded in greatly overcoming the difficulties. He uses a piece of stout platinum wire which is fixed in the bottom of the boiling vessel by means of fusible glass. On account of the superior thermal conductivity of the metal, the boiling is said to take place exclusively from the piece of platinum, and bumping is avoided entirely. I regret to say that I have not yet had an experience with Beckmann's apparatus. In fact, we ordered one from Germany, but on my first attempt to make trial of it, the boiling vessel was found to be cracked at the point of insertion of the platinum. This was merely an accident, but it is an accident which may often occur. A small portion of the bottom of the boiling vessel is thickened by the fusible glass, and there is no wonder if the bottom is cracked by the slightest carelessness in heating.

The means which I adopt in producing a constant temperature in the boiling liquid has already been described in my paper, on "The determination of the temperature of steam arising from boiling salt solutions," and consists in passing a current of the vapour of the

solvent into the boiling liquid or solution, the amount of the vapour thus passed in from without being regulated by the height of the lamp. The degree of constancy of the temperature attained in this manner by the boiling liquid is really astonishing, a most delicate thermometer, capable of showing $\frac{1}{1000}$th of a degree, remaining almost stationary.

Description of Apparatus. Instead of the three-necked flask with a piece of platinum wire fixed into the bottom and with a complex condensing arrangement, used by Beckmann, I employ, for the *boiling vessel*, an ordinary U-tube, A, about 2 centimeters in internal diameter and 21 centimeters in height. Near the top of one limb of this U-tube, a hole is blown out by means of the blowpipe, which serves for jointing on, with a piece of india-rubber tubing, a small side-tube having a glass stopper-cock *s*. The side-tube is, in its turn, connected with an ordinary Liebig's condenser C. The *boiler* for generating the vapour of the solvent is an ordinary round-bottomed flask B, provided with a cork, through which pass the stem of a tapped funnel *f* and a delivery tube *d*. The latter is connected, by means of a piece of thick india-rubber tubing *i*, with a tube *e*, drawn out and slightly bent, at its lower end, so that this end of the tube may just reach the bottom of the boiling vessel, when the cork, through which it passes above, is fixed into the mouth of the U-tube, nearest to the boiler. This tube *e* thus establishes communication between the boiler and the boiling vessel. A tin-plate vessel *pp* with a hole in its bottom and resting upon a tripod serves to enclose the boiler and to protect it from draughts. The boiling vessel is, on the other hand, enclosed by a box made of thick asbestos card-board *qq*, with a movable bottom. When the solvent is of low boiling point, such as carbon bisulphide or ether, a piece of plain asbestos card-board serves as the bottom of the box, and the boiling vessel is heated by

indirect flame, as shown in the figure. In the case, however, of less volatile solvents, such as water or even alcohol, a piece of asbestos card-board having a hole in the centre is used instead, and the boiling is produced by direct flame.

The height of the flame both under the boiler and the boiling vessel has to be carefully regulated for each particular solvent according to its volatility; and, in order to avoid the effect of any alteration in the gas pressure owing to constant openings and shuttings of gas taps in other parts of the laboratory, the following arrangement is adopted and is found to answer admirably. The gas tube is connected with three Bunsen burners and, fully opening the gas tap, the burner nearest the tap is lighted. This acts as a gas regulator. The supply of gas to the other two burners is adjusted by means of screw-clips *k* and *l*; and, since the amount of gas supplied to the regulator-burner is incomparably greater than that supplied to the two others, no effect of alterations of gas pressure in the mains can be perceived in the latter. This circumstance is of considerable importance, because I am thereby enabled to use the bulb of the thermometer always naked, there being no necessity to envelope it with asbestos fibres, as when working with Beckmann's apparatus.

The thermometer which I use includes only a range of 6°c. and is graduated into $\frac{1}{100}$th of a degree. Each of these divisions (about 0.4 mm.) can be readily estimated into $\frac{1}{10}$th by means of a reading telescope, which I always use, so that the thermometer can indicate $\frac{1}{1000}$th of a degree. There is an arrangement in the instrument by which the quantity of mercury in the bulb may be increased or diminished, so that it can be used either for low or high temperatures. This instrument has already been described by Beckmann.

The whole arrangement is exceedingly simple, and can be set up by any one with materials commonly found in all laboratories. This,

I venture to say, is a great advantage. The only costly part of the apparatus is the thermometer, but for good and accurate work in most of the physico-chemical inquiries, a delicate thermometer is indispensable and, after all, its cost is not disproportionate to the important services it can render.

Mode of Working. For working with the apparatus described above, I begin by inserting the tube *e* into one limb of the boiling vessel, fixing it by means of the cork *c*. Small glass beads are then introduced into the boiling vessel through the other limb, till they quite fill the bent portion of the latter. (Beckmann recommends the use of small garnets or glass beads for preventing local differences of temperature, and I find them also very effective for thoroughly mixing the vapour passed in with the boiling liquid and thus establishing a perfect uniformity of temperature.) The experimental liquid is now poured in, so that it occupies a space of 3 or 4 centimeters above the level of the glass beads in the open limb, and the latter is then closed by the cork carrying the thermometer. The lid of the asbestos box is next slipped down the two limbs of the U-tube, which pass through two holes previously bored in it in proper positions, and the boiling vessel, after having been properly enclosed in the asbestos box and supported on the ring of a retort-stand, is connected, on the one side, with the boiler and, on the other, with the side tube and condenser, as shown in the figure.

First the boiler, half filled with the solvent, and then the boiling vessel are heated by carefully regulated lamps, the stop-cocks *r* and *s* both open ; and, when the liquid in both vessels begins to boil properly, the stop-cock *r* is closed, and the two lamps very carefully regulated, so that the height of the column of the boiling liquid remains, as nearly as can be judged, level with the lid of the asbestos box for, at least, a quarter of an hour. This circumstance is used as

the criterion for judging that the boiler and the boiling vessel are each being heated by a properly adjusted lamp, so that evaporation and condensation balance each other in the boiling vessel. The thermometer rises rapidly at first but more slowly afterward and attains the maximum temperature in 10 or 12 minutes, which then remains perfectly constant. The top of the thermometer is now lightly tapped for some seconds, and the observation of the exact temperature made by the aid of a reading telescope. Finally, the volume of the distillate collected in the small flask F is noted and, after withdrawing the lamps, the stop-cock *r* is opened and the stop-cock *s* closed.

I make it a rule to repeat this observation of the exact boiling point of the solvent once or, in some cases, twice more in order to make quite sure of the temperature. For this purpose, the distillate is returned to the boiler and the whole of the above process repeated under exactly the same conditions as before, taking the final observation of the temperature when the distillate reaches the mark previously made on the receiver flask.

The next step is to introduce the substance into the solvent. This may be done in two ways. The first and simplest method, which should be used when the solvent is of low boiling point, consists in taking out the cork, which carries the thermometer, adding the solid substance to the solvent, and then immediately replacing the cork to it original position. The other method, which is employed in the case of solvents of high boiling point and where, therefore, if the first method were adopted, the highly heated thermometer would have to be suddenly exposed to the cold air, is as follows: After the exact determination of the boiling point of the solvent has been made, the boiler is a little cooled without opening the stop-cock *r*, when a small quantity of the solvent runs up the tube *e* and flows down into the boiler. The stop-cock *r* is now opened and then, dis-

connecting the delivery tube *d* from the tube *e*, a small quantity of a concentrated solution of the substance, made beforehand, is introduced into the boiling vessel by means of a small funnel attached to the india-rubber tubing *i*. After removing the funnel, communication is re-established between the boiler and the boiling vessel.

The lamps, which have been put aside but not blown out, are next brought under the two vessels as before, and the boiling point of the solution determined in exactly the same manner and under exactly the same conditions as those in which the boiling point of the solvent was determined, collecting, of course, the same quantity of the distillate. When the exact temperature has been ascertained, the lamps are put out, the stop-cock *r* opened, the stop-cock *s* closed, and the whole apparatus allowed to cool.

After the solution has completely cooled, the percentage composition of the solution is determined by transferring some of the solution into a weighed stoppered bottle by means of a pipette and, after ascertaining the weight of the solution taken, evaporating the latter at a gentle heat on a water bath and weighing the fixed residue. In case of iodine, it is determined volumetrically by a standard solution of sodium thiosulphate.

From the account above detailed of the process I adopt, it will be seen that it differs from the method used by Beckmann essentially in this that, whilst Beckmann takes a weighed quantity both of the solvent and of the solid substance, and determines the boiling points in an apparatus provided with a reflux condenser and, therefore, requires a complicated arrangement, the method here described requires no special apparatus, as it determines the boiling points in the ordinary manner, but obviating the usual difficulties of such determinations and, at the same time, keeping the quantity of the boiling liquid constant, by passing a current of the vapour of the solvent

from without. Another difference which, I venture to say, is also an improvement is that the boiling point of the solution is ascertained just before its composition is determined and it, therefore, renders the connection between these properties more certain than in the case of Beckmann's method. The following tables contain the results of my determinations.

Solvent: *Water.*

Lat. Ht. Vap. = 536.4 Cal. (Regnault 1847). B.P. = 100°C. $\frac{0.02T^2}{W} = 5.2$

Substance.	Boil. Point. Solvt.	Soln.	Rise.	g-Soln.	g-Subst.	g-Solvt.	g-Subst. per 100 g. Solvt.	Obs. Mol. Wt.	Diff. per 100.
Mercuric Chloride: $HgCl_2$ = 271.	2.262	2.441	0.179	10.6755	0.8795	9.7960	8.978	**260.8**	−3.8
,,	2.305	2.389	0.084	10.4385	0.4258	10.0127	4.253	**263.3**	−2.8
Mannite: $C_6H_8(OH)_6$ = 182.	2.214	2.406	0.192	11.2302	0.6855	10.5447	6.501	**176.1**	−3.2
,,	2.265	2.396	0.131	11.7120	0.5196	11.1924	4.642	**184.1**	+1.2
,,	2.403	2.483	0.080	9.5396	0.2637	9.2759	2.843	**184.8**	+1.5
Cane Sugar: $C_{12}H_{22}O_{11}$ = 342.	2.475	2.680	0.205	10.7289	1.2980	9.4309	13.775	**349.4**	+2.2
,,	2.317	2.447	0.130	11.7419	0.9175	10.8244	8.476	**339.0**	−0.9
,,	2.649	2.713	0.064	10.6146	0.4392	10.1754	4.316	**350.7**	+2.5
,,	2.653	2.688	0.035	10.7965	0.2579	10.5286	2.447	**363.5**	+6.3

Solvent: *Ethyl Alcohol.**

Lat. Ht. Vap. = 214.9 Cal. (Regnault 1862). B.P. = 78.°3C. $\frac{0.02T^2}{W} = 11.5$

Substance.	Boil Point.			g-Soln.	g-Subst.	g-Solvt.	g-Subst. per 100 g. Solvt.	Obs. Mol. Wt.	Diff. per 100.
	Solvt.	Soln.	Rise.						
Acetanilide: $C_6H_5.NH.C_2H_3O = 135.$	3.720	4.191	0.471	8.0628	0.4338	7.6290	5.686	138.8	+2.8
,,	3.554	3.705	0.151	8.3318	0.1410	8.1908	1.721	131.1	−2.9
,,	3.793	3.906	0.113	8.2361	0.1090	8.1271	1.341	136.5	+1.1
,,	3.581	3.667	0.086	8.0012	0.0817	7.9195	1.032	138.0	+2.2
Salicylic Acid: $C_6H_4.OH.CO_2H = 138.$	3.984	4.515	0.531	8.0762	0.4772	7.5990	6.280	136.0	−1.5
,,	3.775	4.092	0.317	7.0653	0.2523	6.8130	3.704	134.4	−2.6
,,	4.240	4.367	0.127	8.3855	0.1279	8.2576	1.549	140.3	+1.7
Mercuric Chloride: $Hg\ Cl_2 = 271.$	4.050	4.521	0.471	6.8492	0.6717	6.1775	10.873	265.5	−2.0
,,	3.731	4.111	0.380	8.2052	0.6612	7.5440	8.765	265.2	−2.1
,,	3.702	3.848	0.146	8.4119	0.2774	8.1345	3.413	268.9	−0.8

* Dried by distilling first from freshly ignited lime and then from a small quantity of Sodium.

Solvent: *Ethyl Ether.**

Lat. Ht. Vap. = 90.1 Cal. (Regnault 1862). B.P. = 34.°9C. $\frac{0.02T^2}{W} = 21.0$

Substance.	Boil. Point.			g-Soln.	g-Subst.	g-Solvt.	g-Subst. per 100 g. Solvt.	Obs. Mol. Wt.	Diff. per 100.
	Solvt.	Soln.	Rise.						
Salicylic Acid: $C_6H_4.OH.CO_2H = 138.$	1.013	1.397	0.384	6.9880	0.1733	6.8147	2.543	**139.1**	+0.8
,,	1.337	1.661	0.324	7.2700	0.1591	7.1109	2.237	**145.0**	+5.1
,,	0.986	1.236	0.250	7.1384	0.1174	7.0210	1.672	**140.4**	+1.7
Naphthalene: $C_{10}H_8 = 128.$	0.739	1.464	0.725	7.2743	0.3128	6.9615	4.493	**130.1**	+1.7
,,	1.038	1.541	0.503	7.0014	0.2143	6.7871	3.159	**131.9**	+3.0
,,	1.275	1.542	0.267	7.0232	0.1112	6.9120	1.609	**126.6**	−1.1
Iodine: $I_2 = 254.$	1.295	1.615	0.320	7.1600	0.2678	6.8922	3.886	**255.0**	+0.4
,,	1.098	1.302	0.204	7.2838	0.1796	7.1042	2.528	**260.2**	+2.4
,,	1.174	1.255	0.081	6.9250	0.0692	6.8558	1.009	**261.6**	+3.0

* Redistilled over Sodium.

Solvent: *Carbon Bisulphide.**

Lat. Ht. Vap. = 84.8 Cal. (Regnault 1862). B.P. = 46.°2C. $\frac{0.02T^2}{W} = 24.0$

Substance.	Boil. Point. Solvt.	Boil. Point. Soln.	Boil. Point. Rise.	g-Soln.	g-Subst.	g-Solvt.	g-Subst. per 100 g. Solvt.	Obs. Mol. Wt.	Diff. per 100.
Naphthalene: $C_{10}H_8 = 128$.	3.480	3.860	0.380	7.6888	0.1433	7.5455	1.901	**120.0**	−6.2
,,	3.385	3.718	0.333	12.4832	0.2192	12.2640	1.795	**130.5**	+1.9
Iodine: $I_2 = 254$.	3.887	4.284	0.397	11.3537	0.4714	10.8823	4.332	**261.9**	+3.1
,,	3.680	3.905	0.225	11.4810	0.2600	11.2210	2.317	**247.1**	−2.7
,,	3.709	3.892	0.183	7.9370	0.1534	7.7836	1.971	**258.5**	+1.8
Sulphur: $S_8 = 256$.	3.551	4.141	0.590	10.5998	0.6247	9.9751	6.263	**254.8**	−0.4
,,	3.430	3.791	0.361	12.2753	0.4488	11.8265	3.795	**252.3**	−1.7
,,	3.682	3.870	0.188	10.2030	0.1998	10.0032	1.997	**254.9**	−0.4

* Purified by shaking with mercury and then distilling from dry litharge.

In conclusion, a word may be said with regard to the molecular weights of iodine and of sulphur in solution. Paternò and Nasini (*Ber. d. Deutsch. Chem. Ges.*, **21**, 2153 [1888]), by the method of the freezing point depression, found that the molecular weight of iodine in very dilute benzene solutions is represented by I_2, whilst it possesses higher values in more concentrated solutions. They found also that iodine is more or less dissociated into the atomic state in its solution in glacial acetic acid. Loeb (*Zeit. physik. Chem.*, **2**, 606 [1888]), however, could not verify these conclutions by the same method; and, both on account of the small solubility of iodine in either benzene or glacial acetic acid and of the great variations in the results according to concentration, he ultimately gave up these experiments. By determining the vapour pressures of solutions of iodine in ether and in carbon bisulphide, Loeb showed, on the other hand, that the molecular magnitude of iodine in brown etherial solutions is represented by I_4, whilst in carbon bisulphide it is less complex and corresponds to the formula $I_{2.4}$. Lastly, Beckmann (*Zeit. physik. Chem.*, **5**, 76 [1890]) found, by the aid of his boiling method, that no appreciable difference exists in the molecular magnitude of iodine, whether dissolved in ether or in carbon bisulphide, and that it corresponds to I_2 in both of the solutions. The numbers he obtained varied between 235 and 256 in etherial solutions, and between 250 and 272 in carbon bisulphide solutions ($I_2 = 254$). More recently, Hertz (*Zeit. physik. Chem.*, **6**, 358 [1890]), by the freezing method, obtained numbers varying between 255.5 and 276.0 for the molecular weight of iodine in its dark red solution in naphthalene, and pointed out that the red colour of the iodine solution is not due to the existence of complex molecules, as already shown by Beckmann. My own determinations also fully confirm the results obtained by the latter.

With regard to the molecular weight of sulphur in carbon bisulphide solution Beckmann, by the boiling method, obtained numbers varying between 245 and 281 and, therefore, pointing to the existence of complex sulphur molecules represented by $S_8=256$. The formula S_6 had generally been regarded as the maximum molecular formula of sulphur, and the results obtained by Beckmann, therefore, needed confirmation. The experiments by Hertz (*loc. cit.*) who obtained numbers varying between 262.3 and 279.4 and, still more conclusively, my own determinations, put Beckmann's views beyond any doubt.

A Simple Experiment in Chemical Kinetics.

by

K. Ikeda, *Rigakushi.*

The usual lecture experiment of absorbing oxygen from air by a piece of phosphorus presents a very simple case of chemical kinetics. For, if the lump of phosphorus employed be not too small, the active surface remains practically constant, and the chemical action must be proportional to the density of the oxygen left. The course of the chemical change can be followed by observing the diminution of pressure or of volume. The chief disturbing effect is the elevation of temperature consequent on the oxidation, but this can be prevented to a great extent by keeping the phosphorus cool.

To this end a sufficient quantity of pure clear phosphorus was melted between two test-tubes; the inner tube was next filled with cold water when phosphorus solidified and could be easily drawn out of the external tube. In this manner a uniform cylinder of phosphorus was formed around the lower part of the smaller test-tube. Any irregularities in surface of the phosphorus were removed with a sharp knife, and the whole carefully smoothed. The phosphorus cylinder thus prepared was about 30 m.m. long, a little less than 1 m.m. thick, and 12 m.m. in external diameter, so that the total surface was very nearly 11 square c.m. To keep this cylinder cool, the test-tube was fitted with

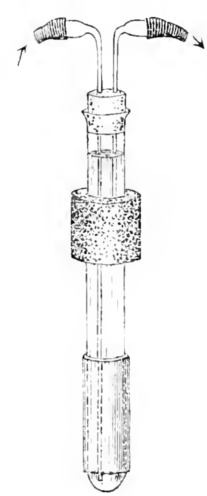

Fig. 1.

a cork furnished with two thin tubes, through which a current of water at about 19°C. was maintained. The annexed figure shows the phosphorus cylinder thus equipped. The surface remained perfectly smooth to the end of the experiment, even some thin striæ left by the knife remaining perfectly distinct. The diminution of phosphorus during the experiment was, of course, very slight, in the second case not amounting to more than one-hundredth of a m.m. in thickness.

The first arrangement.

A large two mouthed bottle (1.5 litre capacity) containing air and a layer of water, was surrounded with water, as shown in fig. 2. The side mouth was closed with a caoutchouc stopper holding a manometer and a short glass tube with india-rubber tube and pinch-cock. The phosphorus tube was now introduced into the bottle through the central opening which it closed air-tight by means of a stopper surrounding its upper part. To make the air-tightness certain, the mouths were both covered with water. Before beginning the observation, the pinch-cock was opened for a moment to equalize the internal and external pressures. As oxidation proceeded the internal pressure gradually diminished, and the reading of the manometer repeated every five minutes gave the following numbers :—

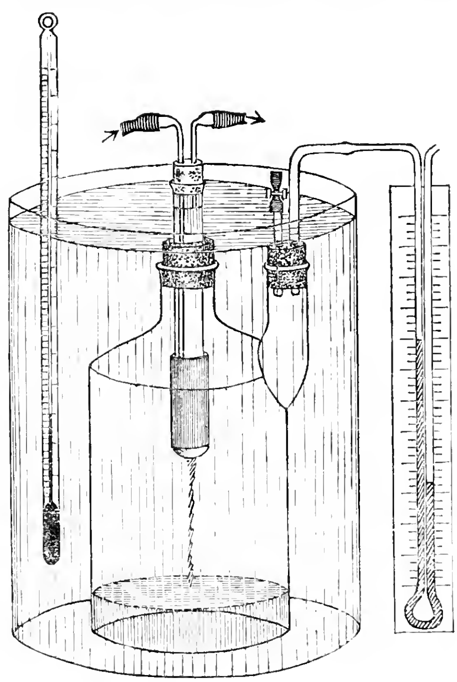

Fig. 2.

Time.	Diminution of pressure.	Pressure of Oxygen left.	$k=\frac{1}{\vartheta}\log\frac{A}{A-x}$.
ϑ	x	$A-x$	
0′	0.0	155.3	———
5′	5.0	150.3	0.00284
10′	10.9	144.4	0.00316
15′	16.6	138.7	0.00327
20′	21.5	133.8	0.00324
25′	26.5	128.8	0.00325
30′	31.1	124.2	0.00324
35′	36.3	119.0	0.00330
40′	40.7	114.6	0.00330
45′	44.9	110.4	0.00330
50′	49.1	106.2	0.00330
55′	53.5	101.8	0.00333
60′	57.3	98.0	0.00333
65′	61.3	94.0	0.00335
70′	65.5	89.8	0.00340
75′	69.7	85.6	0.00345
80′	73.8	81.5	0.00350
∞	155.3	0.0	———

A, the initial pressure due to oxygen, was not directly measured but calculated out in the following manner :—

A = (760 m.m. — 17. m.m.) 0.209 = 155.3 m.m.

Where 760 m.m. : Barometric height,

17 „ „ : The vapor-tension of water, the temperature of the surrounding water being 20.°0 — 20.°1 during the experiment.

20.9 % : the percentage of oxygen in air.

As the volume of air remained constant, the active mass of oxygen at time ϑ is proportional to $A-x$, so that the equation $\frac{dx}{d\vartheta}=k(A-x)$ ought to be satisfied in this case. But $\frac{\Delta x}{\Delta\vartheta}$ did not diminish regularly as the time extended, and therefore $\frac{1}{\vartheta}\log\frac{A}{A-x}$

is not constant as it should be, but grows gradually larger. This may be owing to the heating effect of oxidation, or to some other cause. For one thing the manometer being hastily extemporized did not allow of any exact reading. On the whole, the experiment was a hasty and rough one, and the result must not be regarded as final. Still the general agreement with the law of mass action is apparent.

With a strong solution of mercuric iodide in potassium iodide solution, or some other heaving liquid properly coloured, as the manometric liquid, this arrangement may be employed in a lecture experiment to illustrate the simple principle of chemical kinetics.

The second arrangement.

A burette B of 100 c.c. capacity and 18 m.m. internal diameter was connected with a water reservoir C and surrounded with water as shown in Fig. 3. The phosphorus cylinder was introduced into the upper part of the burette ; and after adjusting the level of water in B and C, the volume of air in the former was accurately observed which gave, with proper correction, the initial volume $v_0 = 93.5$ c.c. The observation was repeated every five minutes, the pressure in B being always kept equal to that of the external atomosphere, so that the diminution of the volume x here denoted the amount of oxygen absorbed. The result is shown in the following table.

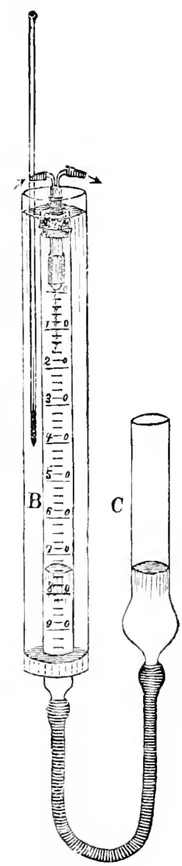

Fig. 3.

Time.	Volume.	Diminution of volume.	Volume of oxygen left.	$k = \frac{1}{\vartheta}\left\{ \frac{(x + v_0 - A)}{\log \frac{A}{A-x}} \right\}$.
ϑ	v	x	$A-x$	
0′	93.5	0.0	17.5	——
5′	91.9	1.6	15.9	1.78
10′	90.5	3.0	14.5	1.73
15′	89.5	4.0	13.5	1.58
20′	88.3	5.2	12.3	1.60
25′	87.3	6.2	11.3	1.58
30′	86.3	7.2	10.3	1.58
35′	85.4	8.1	9.4	1.58
40′	84.6	8.9	8.6	1.58
45′	83.8	9.7	7.8	1.58
50′	83.1	10.4	7.1	1.58
55′	82.5	11.0	6.5	1.57
60′	81.9	11.6	5.9	1.57
65′	81.4	12.1	5.4	1.56
70′	80.9	12.6	4.9	1.56
75′	80.4	13.1	4.4	1.58
80′	80.0	13.5	4.0	1.57
85′	79.6	13.9	3.6	1.58
90′	79.3	14.2	3.3	1.57
95′	79.0	14.5	3.0	1.56
∞	76.0	17.5	0.0	——

During the experiment the temperature of the surrounding water did not vary more than three-tenths of a degree (20.°5—20.°8) and as this affects only the second place of the decimals in the reading of volumes, no correction has been introduced for it. The temperature of the water current passed through the test-tube remained constant at 18.°5. The final volume of air or rather nitrogen was observed after the lapse of 48 hours and gave on proper correction 76.0 c.c. This subtracted from the initial volume gave $A = 93.5 - 76.0 = 17.5$ c.c. for the initial volume of oxygen.

In this case the volume of air diminished gradually, and the chemical action at time ϑ was proportional not to $A-x$, but to $\frac{A-x}{v}$, so that

$$\frac{dx}{d\vartheta}=k\frac{A-x}{v}=k\frac{A-x}{v_0-x}$$

or
$$kd\vartheta=\frac{v_0-x}{A-x}dx=\left(1+\frac{v_0-A}{A-x}\right)dx$$

which on integration between o and x gives

$$k=\frac{1}{\vartheta}\left\{x+(v_0-A)\log\frac{A}{A-x}\right\}$$

Here it is, of course, necessary to employ the natural logarithms.

As the last column in the foregoing table shows, $\frac{1}{\vartheta}\left\{x+(v_0-A)\log\frac{A}{A-x}\right\}$ is practically constant with the exception of the first two numbers. This deviation allows of several explanations and suggests various further experiments; but speculations had better be omitted, especially since there is but the single experiment recorded above to go upon. Still it would be highly interesting to see what influence various diluents have on the velocity of the oxidation, especially as some gases are said to retard the action greatly.

The Chemical Laboratory of the
Higher Normal School, Tokyo.
August, 1892.

Imidosulphonates.

by

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and

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This paper is based upon an investigation of the sodium imidosulphonates and their derivatives, all hitherto unknown, but includes, besides, some new things about the ammonium and potassium salts. Several mixed tribasic salts are described in it, particularly those of lead, of silver, and of mercury, which show peculiarities in their constitution. It also contains the determination by experiment of the nature of sulphatammon (Rose), a compound of one molecule of sulphur trioxide with two of ammonia.

History.—Rose (1834–1840; *Pogg.*, **32**, 81; **47**, 471; **49**, 183). Jacquelain (1843; *Ann. Chim. Phys.*, [3] **8**, 293), Fremy (1845; *Ann. Chim. Phys.* [3] **15**, 408). Woronin (1859; *J. Russ. Ch. Soc.* **3**, 273). Claus and Koch (1869; *Ann. Chem.*, **152**, 336), Berglund (1875; *L. univ. Arsk.*, **12** and **13**; *Bull. Soc. Ch.*, **25**, 455; **29**, 422; *Ber. Ch. Ges.*, **9**, 252 and 1896). Raschig (1887; *Ann. Chem.*, **241**, 161), and Mente (1888; *Ann. Chem.*, **248**, 232) are the chemists who have worked upon the inorganic salts now known as imidosulphonates.* The constitution of these salts has been established mainly through investigations by Rose, Jacquelain, and Berglund.

* Woronin's paper we only know from abstracts in the *Jahresbericht* and Watts's *Dictionary*. Berglund's memoir in the *Lunds universitäts Arskrift* we have also not seen.

Rose discovered in 1834 that the compounds of sulphur trioxide with ammonia, his *sulphatammon* and *parasulphatammon*, give, when freshly dissolved in water, the reactions of a sulphate very incompletely or not at all, and do not yield up all their nitrogen as ammonia to chloroplatinic acid and other reagents.

Jacquelain discovered the then unlikely fact that the stable compound of sulphur trioxide and ammonia, his *sulphamide*, is derived from two molecules of the oxide and three of ammonia.

Berglund determined the constitution of this stable compound to be imidosulphonic, by ascertaining that it behaves as an ordinary ammonium salt only to the extent of two-thirds of its nitrogen.

To Fremy is owing the discovery that hydrolysis of the main product of the action of a nitrite upon a sulphite yields a salt, his *sulphamidate*, now known to be an imidosulphonate ; and that this salt exhibits the reactions of neither nitrite nor sulphite and can be so far hydrolysed as to yield ammonia.

He and, after him, Claus and Koch having attributed to this salt a composition which allowed of its being written down as an addition-compound of ammonia, Berglund proved them to have erred by himself establishing the identity of Fremy's potassium sulphamidate, Claus's *disulphammonate*, with potassium imidosulphonate prepared from ammonium imidosulphonate coming from chlorosulphonic acid and ammonia. Mendeléeff (abs. in *Ber. Ch. Ges.*, **3**, 872) had just before suggested such a constitution as possibly belonging to Fremy's sulphamidates. In his *Principles of Chemistry*, (1891), however, apparently unaware of Berglund's results, he treats parasulphatammon as being ammonium amidosulphonate. More recently, Raschig showed that the properties of Fremy's salts are inconsistent with Claus's views as to their being quinquevalent nitrogen compounds.

Claus recognised the sulphonic constitution of these salts, while

Berglund's discovery that they issue from the interaction both of sulphur trioxide and ammonia, and of sulphur dioxide and a nitrite, supplied a beautiful, if not needed, confirmation of Claus's theory.

Although, as we shall demonstrate, Rose, Jacquelain, Fremy, and Woronin had all prepared imidosulphonates with three equivalents of base, as well as with two, to one of nitrogen, yet no clear conception of these compounds as salts of a more than bibasic acid had been formed before they were examined by Berglund and shown by him to be derivatives of a tribasic acid, one in which the aminic, as well as the sulphonic, hydrogen is basic. He prepared for the first time tripotassium and some other trimetal salts.

It will be seen that the existence of imidosulphonates with three equivalents of base, first made evident by Berglund, constitutes a misnomer the term *imidosulphonate*, which he himself introduced, since in these salts the third hydrogen of ammonia is gone. But no simple way out of the difficulty presents itself, and we adopt it, therefore, as Raschig and Mente have done before us.

Imidosulphonic Acid.

Imidosulphonic acid, $HN(SO_3H)_2$, has been obtained only in solution. It was first prepared by Jacquelain. We have followed his process, which consists in treating the lead salt in water with hydrogen sulphide, and made it our special care to have the lead salt as free as possible from sodium. The lead salt was decomposed by the hydrogen sulphide with a very sensible elevation of temperature, and this strengthened our expectation that the acid must hydrolyse almost as fast as it forms. But such is not the case, and rapid filtration over the pump gave a concentrated solution containing very little sulphuric acid, and therefore very little amidosulphonic acid.

The imidosulphonic acid was fully identified by its giving an

abundant precipitate with excess of baryta water, freely soluble in nitric acid, exclusive of a little sulphate; and particularly by its yielding a crystalline precipitate, also abundant, with potassium acetate. Jacquelain converted his preparation into the very soluble dibarium salt by just neutralising the acid with the baryta, knowing, however, that excess would give him the insoluble salt. On leaving the solution of the acid in a vacuum-desiccator it was found by us to have almost wholly hydrolysed into amidosulphonic acid in eighteen hours or less.

Fremy attempted to prepare the acid by adding hydrofluosilicic acid to potassium imidosulphonate and found it to decompose almost immediately. He made out the products of its decomposition to be sulphuric acid, sulphurous acid, and ammonia, but in this he must have erred. Berglund prepared the acid as Jacquelain had done, and found it very unstable.

We find that dry dipotassium imidosulphonate may be mixed with dry potassium hydrogen sulphate, and fused with it without being affected. At temperatures high enough, 420°–450°, the imidosulphonate suffers the decomposition proper to it when alone. To observe the inactivity of the salts ordinary moist air has to be excluded, and we therefore worked in a Sprengel-pump vacuum. From the fused mixture of the two salts, cooled and powdered, alcohol dissolves out only a very little sulphuric acid; ether does the same.

Dipotassium imidosulphonate, in dry, fine powder, mixed with two molecular proportions of sulphuric acid, sp. gr. 1.84, forms a thin paste, which kept in a dry atmosphere soon solidifies to a hard, somewhat translucent mass. Powdered and exposed to air, this mass does not deliquesce. Left for days in dry air it experiences no evident change. On dissolving such a preparation in water, to which a little potassium hydroxide has been added to guard against much hydrolysis of any component, it yields much imidosulphonate unchanged,

together with amidosulphonate, but no ammonia. The amidosulphonate would no doubt be absent but for water taken up from the air during mixing, from the sulphuric acid itself, and lastly, to some degree, during dissolution in the water in spite of the alkali present. When heated, the mass very readily fuses and gives off sulphuric acid and sulphur dioxide. Jacquelain observed similar behaviour with sulphuric acid, hot and cold, in diammonium imidosulphonate.

The little action of concentrated sulphuric and nitric acids was noticed by Fremy, and by Claus and Koch. Evidence, therefore, has not been obtained of any action of sulphuric acid upon a dry imidosulphonate such as would be expressed by the equation.—$HN(SO_3K)_2 + 2SO_4H_2 = HN(SO_3H)_2 + 2SO_4HK$.

Ammonium imidosulphonates.

Diammonium imidosulphonate.—Diammonium imidosulphonate, $HN(SO_3Am)_2$, can be prepared from ammonia and sulphur trioxide (Rose, Jacquelain, Woronin), or chlorosulphonic acid (Berglund, Mente), or sulphuryl chloride or pyrosulphuryl chloride (Mente). It is obtained when nitrous gas is passed into an ice-cold solution of ammonium sulphite, and the product hydrolysed (Fremy). Along with ammonia, it is the product of heating ammonium amidosulphonate to 160° (Berglund).

The decomposition of barium imidosulphonate by a solution of ammonium sulphate can only be very imperfectly accomplished, but by treating it first with dilute sulphuric acid barely sufficient to convert it into the soluble barium hydrogen imidosulphonate and then adding ammonium sulphate until no barium remains in solution, diammonium imidosulphonate can be prepared satisfactorily, except for the incumbrance of the bulky barium imidosulphonate and sulphate. Another procedure which can be followed is to decompose either of the

hydroxy-lead imidosulphonates with a solution of ammonium carbonate. Triargentum imidosulphonate (p. 93) stirred with a solution of its equivalent of ammonium bromide has been our source of the diammonium salt. Triammonium salt is thus produced, and the solution decanted from the silver bromide needs to be evaporated on a water bath until ammonia almost ceases to escape, and then to be filtered from a little silver bromide. To crystallise the salt the solution generally requires further evaporation and, in order to guard against hydrolysis into amidosulphonate during the operation, is kept alkaline by occasionally adding a drop of strong ammonia-water.

Diammonium imidosulphonate forms monosymmetric prisms, isomorphous with the potassium salt (Münzing and Raschig). In its general reactions it is like the sodium and potassium salts. It reddens blue litmus, even while still quite free from sulphuric acid (Jacquelain).

The effect of heating it was tried by Rose, and is described in the first of his papers upon the sulphatammons. According to him the products differ little from those coming from heating ammonium sulphate, except in the absence among them of water. Ammonia and sulphur dioxide are evolved, first the former, then principally the latter. Acid ammonium sulphate makes the residue, at not too high a temperature. In the retort-neck ammonium sulphite and sulphate are formed; in the receiver the former salt only. He also got a little of the yellow compound of the same gases. When the salt is heated in ammonia gas, he found that but little sulphate is obtained.

This account of Rose's of the effect of heating diammonium imidosulphonate is essentially incomplete, but is true to the extent that ammonia first, and then sulphur dioxide are evolved, that these two partly condense as a sublimate, and that acid-sulphate is formed. Heating in a roomy retort, and thus permitting the action of moist

air, favours the change here described, and brings it still nearer to that caused by heat in ammonium sulphate.

Jacquelain found it fusible without decomposition and capable of being kept in fusion without change in a current of dry ammonia, and that, a little above its melting point, it decomposes into ammonia, ammonium sulphite which sublimes, and ammonium acid-sulphate as a residue.

We have found matters otherwise. Diammonium imidosulphonate, heated in a vacuum, suffers no change (except that it yields a very minute liquid sublimate at 190°) until very near to 357°, when it melts and also effervesces or boils. Most of the vapours condense as unchanged imidosulphonate just above the heated part of the tube, but ammonia of very low tension is given off. At last, inconsiderable sublimates, white and yellow, appear some distance up the tube, consisting of compounds of ammonia with sulphur dioxide, and then with the formation of these, minute quantities of nitrogen escape. Stopping the operation in an hour or two, when the evolution of gas is hardly noticeable, the residue is found to be unchanged imidosulphonate, mixed with a little acid-sulphate. When the salt is subjected to a stronger heat, so as to be kept in violent ebullition, most of the unchanged salt condenses and flows back, and but little of it is got as a sublimate. On the other hand, the sulphite-ammonia sublimates are more considerable, though still not abundant, and nitrogen is now given off more freely, but also, as well as free ammonia, still in small quantities, even though the heating be maintained for a long time. In one experiment, after about an hour's violent ebullition, the residue was found, by dissolving it in sodium-hydroxide volumetric solution in a closed vessel and titrating with acid, to contain about one-fifth of its weight of ammonium pyrosulphate. From what precedes it is evident that diammonium imidosulphonate is a comparatively stable body,

boiling and condensing unchanged for the most part. Very slowly, however, even when more strongly heated, it decomposes into nitrogen, ammonia, water, sulphur dioxide, and sulphur trioxide, thus :—

$$6HN(SO_3Am)_2 = 2N_2 + 14NH_3 + 6OH_2 + 6SO_2 + 6SO_3\,,$$

these products then partly condensing and yielding finally—

$$2N_2 + 2NH_3 + 3S_2O_5Am_2 + 3O(SO_3Am)_2.$$

The little ammonia first given off without nitrogen is no doubt attributable to the action of moisture in the somewhat deliquescent salt. This water would cause the hydrolysis of its molecular equivalent of the salt into amidosulphonate and acid-sulphate, and then, by further heating, into pyrosulphate and ammonia :—

$$HN(SO_3Am)_2 + H_2O = H_2NSO_3Am + HOSO_3Am = H_3N + O(SO_3Am)_2$$

Experiments, as yet unpublished, by one of us have fully established the existence of the undescribed salt, $O(SO_3Am)_2$, ammonium pyrosulphate.

Triammonium imidosulphonate (hydrated).—This salt, hitherto undescribed, is formed by the union of ammonia with diammonium imidosulphonate in the presence of water. The ammonia must be in excess as else the salt is partly decomposed by water. The most concentrated ammonia-water precipitates it from a saturated solution of the diammonium salt ; also from a concentrated solution of the disodium salt, but then it is mixed with sodium ammonium salt (p. 75) Ammonia-gas acts similarly and much more effectively. Thus obtained, it forms a crystalline powder, but it can be got in large crystals by dissolving it, or the diammonium salt, to saturation in moderately strong ammonia-water, in a closed vessel by the aid of heat, and setting the solution aside to cool. It is also got, and in the largest crystals, by evaporating its ammoniacal solution in a desiccator,

over solid potassium hydroxide and in a strongly ammoniacal atmosphere.

The crystals resemble those of the tripotassium salt and are, therefore, asymmetric. They are clear and lustrous, but gradually effloresce (mostly through loss of ammonia) and become opaque and damp. The salt when dry smells mildly of ammonia, and its crystals in a dry ammoniacal atmosphere over solid potasssium hydroxide very slowly become anhydrous and opaque throughout, while retaining much of their lustre and hardness.

Heated, the salt becomes, mainly, the diammonium salt, but to some extent it melts, loses both ammonia and water, and hydrolyses into sulphate.

Dissolved in water it decomposes, but very incompletely, into the diammonium salt and ammonia, and the solution evaporated in the open air yields crystals of only the diammonium salt. Hence, when either the trisodium or the tripotassium salt is mixed with ammonium chloride, much ammonia is liberated.

It was analysed sufficiently by heating it to 160° very gradually in a current of dried air, in order to determine the water and third atom of ammonia. The sulphur was also estimated, and the salt was thus proved to have a composition analogous to that of the potassium salt, namely, $AmN(SO_3Am)_2, OH_2$. When analysed the crystals were damp but transparent, after rapid pressure in filter paper and weighing.

	Calc.	Found.
Sulphur	26·0	25·4
Loss at 160°	14·2	16·9

Anhydrous triammonium imidosulphonate. Sulphatammon.—Anhydrous triammonium imidosulphonate is, as we shall show, Rose's sulphatammon. Rose found that dry ammonia conducted into a flask

lined with sulphur trioxide combines with the trioxide yielding two products, one a hard vitreous matter replacing the lining of sulphur trioxide, the other a loose flocculent deposit. Jacquelain got the same flocculent body when he mixed vapour of sulphur trioxide with dry ammonia in excess. Rose found that both the flocculent and vitreous matters when dissolved in water and evaporated yielded crystals of the same substance, but that the vitreous matter was often acid through, as he considered, imperfect action of the ammonia. The crystals from solution he called *parasulphatammon*: they are now known to be diammonium imidosulphonate, and from what we ourselves have seen of sublimed ammonium imidosulphonate, we have very little doubt of the vitreous matter being the same salt, and of its occasional acid reaction being due to its hydrolysis. The crystals are therefore formed from three volumes of ammonia and two of sulphur trioxide, but Rose made them out to have the same composition as the undissolved flocculent matter, and to be formed therefore from two volumes of ammonia and one of sulphur trioxide. He recognised, however, such differences in their properties that he held them to be distinct bodies, and called the flocculent matter *sulphatammon*. Jacquelain prepared parasulphatammon from the flocculent matter by two methods, both without the use of water. He found this matter to be variable in composition, and one of his methods of getting crystalline pure parasulphatammon from it, was to fuse it in a current of ammonia, let it solidify, and then stop the further entrance of ammonia before it cooled. His other method was to expose the flocculent matter, first, over sulphuric acid in a vacuum till it ceased to lose weight, and then to a moderate heat, which caused a further loss.

Having in a most careful manner proved the strange erroneousness of Rose's analysis of parasulphatammon, Jacquelain himself made a no less strange misconstruction of the facts he had correctly observed.

He had found that parasulphatammon, as it cooled after being prepared by heat, took up ammonia if in contact with it, and that the flocculent matter, named sulphatammon by Rose, would lose from six to nine per cent. of its weight. in a vacuum over sulphuric acid, leaving a residue giving up more ammonia when heated to 100,° and then consisting of parasulphatammon, as already mentioned. But he treated these phenomena as a case of the physical absorption of gases by porous solids, and likened the ammonia lost or taken up to hygroscopic moisture. Yet a compound of two molecules of sulphur trioxide with four of ammonia has to lose as ammonia only seven and a half per cent. of its weight in order to become a compound of two molecules of sulphur trioxide with three of ammonia. As for the high loss of nine per cent. recorded, that if correctly observed may have been due to loss of moisture, for, as Rose found, sulphatammon is markedly hygroscopic and loses the moisture again at 100.°

Sixteen years later, Woronin first established the accuracy of Rose's analysis of the flocculent matter, sulphatammon, and then showed that this and parasulphatammon were different salts of the same acid. Another sixteen years later, Berglund claimed for sulphatammon that it must be triammonium imidosulphonate, since parasulphatammon had proved to be diammonium imidosulphonate. But then, opposed to this conception of its nature there were the facts, first, that Rose had made no admission of its evolving ammonia when dissolved in water and evaporated so as to yield parasulphatammon; secondly, that Raschig much more recently had dissolved an old specimen of parasulphatammon in ammonia-water and on evaporation over sulphuric acid got only parasulphatammon again ; and, lastly, known to ourselves at least, that trisodium, or tripotassium, or triargentum imidosulphonate when decomposed by ammonium-chloride solution sets free much ammonia. It seemed to us, therefore, at first to be as

evident as it was probable that imidosulphonic acid could not fix a third atom of ammonia along with its imidic hydrogen, and that Berglund's view might not be right.

Experiment has, however, established that sulphatammon is, as Berglund considered it, the triammonium salt. Crystals of the diammonium salt were coarsely ground (they are somewhat hygroscopic, as just stated, and slip under the pestle, and so are not readily ground fine). Of this powder a quantity, equal to 2·29 grammes if dried at 100°, was exposed to a current of carefully dried ammonia, of which it absorbed in two hours $5\frac{3}{4}$ per cent., and in two hours more an additional quantity amounting altogether to $6\frac{1}{4}$ per cent., the temperature being 20.° The calculated quantity to be absorbed is 8 per cent., but considering the coarseness of the particles and the temperature, the result obtained is sufficient. The absorption of the ammonia was attended with a very considerable change in the volume of the solid, and gave it the form of a loose, non-coherent, amorphous powder. Its odour was only mildly ammoniacal, and it was easily preserved without care in a bottle. Dissolved in water it became strongly ammoniacal, and the solution gave a copious precipitate with barium chloride and otherwise behaved as a solution of the crystalline, hydrated salt, which salt indeed it yielded when treated with ammonia gas. Evaporation of the solution gave crystals of the diammonium salt, which is just what Woronin observed in the behaviour of sulphatammon which thus became parasulphatammon.

Exposing it in a desiccator over sulphuric acid for three days at a temperature of or about 20,° we found it to lose only about a half per cent. of its weight so that probably the pure dry salt loses ammonia only in a damp atmosphere, although indeed Jacquelain has found sulphatammon (probably damp) sometimes give up much am-

monia *in vacuo* over sulphuric acid. Having observed the loss of ammonia to be so slight at common temperatures, we heated the same portion in a current of well-dried air for $1\frac{3}{4}$ hours at a temperature of 100–120,° and then found the issuing air to be still carrying ammonia. The salt cooled and reweighed had now lost only 5·4 per cent. in weight by this heating and gave with water a solution which by tests showed very fully the presence of some triammonium salt still.

From what precedes it is evident that triammonium imidosulphonate is comparatively stable in the anhydrous state, and that Rose's sulphatammon is this salt. Coming, as here described, from another imidosulphonate without hydrolysis, it cannot be ammonium amidosulphonate, as it is commonly assumed to be: (as in Mendeléeff's *Principles of Chemistry*, and Ramsay's *System of Inorganic Chemistry*). Further, its conversion by heat into parasulphatammon and ammonia affords no proof that it is ammonium amidosulphonate, for this salt would only change thus at temperatures above 160.° A salt which is certainly ammonium amidosulphonate can be prepared by processes which include the hydrolysis of an imidosulphonate, and this is quite a different body. According to Berglund, it crystallises in large plates, and only at 160° is converted into imidosulphonate and ammonia. It is not readily decomposed by water, is neutral in reaction, and does not precipitate barium salts, facts verified for us by our pupil, Mr. Y. Ōsaka.

Imidosulphonamide.—Mente has found that imidosulphonamide, $HN(SO_2NH_2)_2$, is produced by the action of ammonia upon pyrosulphuryl chloride. Diammonium imidosulphonate, when heated to its subliming point, may possibly yield it, but as at common temperatures it reacts at once with water to form diammonium imidosulphonate, its presence can not be tested for.

Potassium imidosulphonates.

Dipotassium imidosulphonate.—This salt was obtained by Fremy by the hydrolysis of potassium nitrilosulphonate, and this by the action of sulphite upon nitrite. Claus and Raschig have both published accounts of the preparation of the salt in this way. It has also been obtained by double decomposition between diammonium imidosulphonate and a potassium salt (Fremy, Berglund): also from the diammonium salt by Woronin, by converting this first into Jacquelain's barium salt and then decomposing that with potassium sulphate.

Imidosulphonates are derived from nitrilosulphonates by hydrolysis readily enough, but as the two potassium salts are but sparingly soluble in water, and the imide salt almost as readily passes into the amide salt by further hydrolysis, the preparation of dipotassium imidosulphonate as hitherto carried out has proved of uncertain productiveness. On the other hand, the conversion of sodium nitrilosulphonate into disodium imidosulphonate can be very easily and exactly effected, and, since the disodium imidosulphonate is very soluble, the insoluble potassium salt can be prepared from it by double decomposition and precipitation without any trouble. Thus prepared it is also very pure and, consequently, stable. It can also be recovered from mercuric dipotassium imidosulphonate by the action of nitric acid (p. 96)

Dipotassium imidosulphonate results also from the action of heat upon potassium amidosulphonate. The first knowledge of this interesting change—$2H_2NSO_3K = H_3N + HN(SO_3K)_2$—is due to Berglund. We ourselves find that at or just below 350° this change proceeds readily and productively, the gas evolved being ammonia unmixed with any nitrogen. At this temperature the salt, in our

experiments, fused, and then gradually solidified as the decomposition proceeded. The salt used was not, however, pure, being mixed with potassium sulphate.

The effect of heating dipotassium imidosulphonate has been studied by us. Not below the softening point of soft lime-glass does the dry dipotassium salt, in a dry atmosphere, suffer any notable change. In a Sprengel-pump vacuum it evolves gas only very slowly and of very small tension, and yields besides only some very slight sublimates between 360° and 440.° At the temperature of its actual decomposition it melts and boils. The decomposition which then occurs is expressed by the equation—

$$3HN(SO_3K) = NH_3 + N_2 + 3SO_2 + 3SO_4K_2$$

The ammonia and some of the sulphur dioxide condense to form a rather volatile yellow-brown pseudo-sublimate and a less volatile white pseudo-sublimate, both of which readily yield ammonia and sulphur dioxide again when tested in water. The brown sublimate gives a clear and colourless solution. Both sublimates appear to be bodies obtained by Rose from sulphur dioxide and ammonia. Probably, from what Mente has done, the yellow body is nitrogen sulphide and the white is sulphamide. The permanent gases consist of one measure of nitrogen to somewhat more than two of sulphur dioxide. We had expected to be able to get evidence of the intermediate formation of nitrilosulphonate—$3HN(SO_3K)_2 = H_3N + 2N(SO_3K)_3$—by arresting the decomposition before it had gone very far, and by careful regulation of the temperature, but we could not do so, any more than in the case of the diammonium salt.

Fremy observed the formation by heat of potassium sulphate, sulphur dioxide, and ammonia, and of the coloured sublimate. Claus and Koch recorded the decomposition of the salt above 200° sharply into the same bodies. But neither chemist mentions the still more

abundant white sublimate and, what is of more significance, the nitrogen. Had either done so, it would then have been seen that the salt is not an addition compound of ammonia, as they supposed.

One of the remarkable properties we find imidosulphonates, as also oximidosulphonates, to possess, is that of hydrolysing at a sufficiently high temperature by fixing the moisture of the air. Claus had observed slow increase in weight of potassium nitrilosulphonate and in oximidosulphonate (his trisulphammonate and disulphydroxyazate) when they were heated in the air, but wrongly ascribed this increase to oxidation. Oxidation it could not be, for this would only give rise to volatile products, such as water, or oxide of nitrogen, or nitrogen itself, and not increase the weight of the fixed matter. We have, besides, fully proved by experiment that at the temperature, or at a higher one than that even, at which any imidosulphonate increases in weight in ordinary air, that salt remains for hours unchanged in weight and appearance in a current of dried air, and yet is ready, when exposed at the same temperature to undried air, to begin increasing in weight. By the hydrolysis in ordinary air an imidosulphonate loses crystalline lustre, cakes together, and becomes acid. Salts, such as the disodium imidosulphonate, which contain water of crystallisation, require slow drying at a gently rising temperature to prevent hydrolysis, and even then are difficult to dehydrate. Dipotassium imidosulphonate, the crystals of which are anhydrous, is particularly stable and may be heated for hours to 140° or higher in undried air, without the least sensible change. But at 170–180° it slowly fixes water and hydrolyses.

As the tripotassium imidosulphonate is much more soluble than the dipotassium salt the latter dissolves readily in potassium hydroxide solution, from which it can be precipitated by carbonic acid (Raschig). Other and new reactions of the dipotassium salt, observed

by us, are so similar to those we record of the sodium salt, as to make it profitless to give particulars of them.

Tripotassium imidosulphonate.—Tripotassium imidosulphonate was prepared by Berglund, and again by Raschig, by dissolving the dipotassium salt in solution of potassium hydroxide, evaporating, and crystallising.

Sodium imidosulphonates.

Disodium imidosulphonate.—The preparation of this salt depends upon the same changes as those which give the potassium salt by Fremy's method, but the details of working differ materially. Sodium nitrite, as nearly pure as possible, and sodium carbonate in crystals, in the proportion of two molecules of nitrite to three molecules (six equivalents) of carbonate are, with twice their weight or more of water, treated freely with sulphur dioxide in a capacious flask, better fitted with cork and gas tubes. After the carbonate crystals have dissolved, and acid-carbonate has been precipitated and redissolved with continuous effervescence, the liquor, still quite alkaline to litmus, begins to grow warm when more sulphur dioxide enters it. To prevent any considerable rise of temperature, which might cause premature hydrolysis and spoil the operation, the flask is, from this point, kept in motion in cold water, and the stream of sulphur dioxide somewhat lessened and held under watchful control. Neutralisation of the alkaline liquor by the sulphur dioxide entering it is markedly retarded by the action of the nitrite, in the presence of which neutral sulphite cannot continue to exist. When, at length, neutrality to lacmoïd paper is almost reached, the flask is actively agitated without intermission, and the flow of gas first made quite

slow and then at once stopped when the paper is just permanently reddened. The liquor at this stage contains, besides a very little sulphate, nothing but the two salts, nitrilosulphonate and metasulphite, which, omitting notice of intermediate stages, have been derived as follows :—

$$2NaNO_2+3Na_2CO_3+8SO_2=2N(SO_3Na)_3+Na_2S_2O_5+3CO_2$$

The apparently redundant metasulphite is necessary in order to get the nitrite converted wholly into nitrile salt. Its presence is almost equally essential for the preservation from destruction of the imidosulphonate about to be formed.

When full neutrality is reached, or perhaps just passed—it is impossible to say—the liquor after standing a few minutes suddenly becomes strongly acid and warm, and evolves much sulphur dioxide. Hydrolysis has occurred, but thanks to the metasulphite present, only through one stage, and the liquor now holds only imidosulphonate, sulphate, and sulphur dioxide—

$$2N(SO_3Na)_3+Na_2S_2O_5+OH_2=2HN(SO_3Na)_2+2Na_2SO_4+2SO_2.$$

Sulphurous acid appears to have no immediate hydrolysing action upon the imidosulphonate, but as it oxidises to sulphuric acid it must be removed. A rapid current of air is, therefore, sent through the liquor for half an hour or so, after which the addition of a very little sodium carbonate suffices to render it slightly alkaline and quite stable. It is evaporated, so far as may be necessary, on the water-bath and then cooled down to 0° or a little below, whereby in the course of some hours most of the sulphate is made to crystallise out. A second evaporation and cooling is generally needed to separate still more sulphate, after which renewed evaporation and ordinary cooling brings about the separation in well-formed crystals of much of the disodium

imidosulphonate. The mother-liquor can be further worked for sulphate and imidosulphonate, if worth the trouble.

The reaction proves to be almost quantitative, for the salt obtained in crystals is 80 per cent., while there is considerable loss of salt in the mother-liquors adhering to the very voluminous masses of sodium sulphate. The crude salt can be purified by recrystallising from warm water rendered slightly alkaline as a precaution against hydrolysis.

Instead of sodium carbonate the equivalent quantity of normal sulphite, or of metasulphite, pure and therefore freshly prepared, may be used along with the additional sulphur dioxide indicated by the equation already given. The mixture of nitrite and metasulphite becomes at once alkaline and rapidly heats up, so as to need cooling, in accordance with what has been stated. This mode of procedure is certainly less simple and less practicable than that described. It may also be mentioned that a sufficient excess of metasulphite will of itself convert all nitrite to nitrile salt, but as then addition of another acid is required to effect hydrolysis into imidosulphonate the liquor obtained is unsuited to give a good yield of imidosulphonate crystals.

It is not unfrequently convenient to prepare disodium imidosulphonate from the trisodium salt. The latter salt can be preserved without need of taking any precautions, which is not the case with the disodium salt. Again, where the direct process for preparing sodium imidosulphonate has been carried out with impure salts, or with imperfect observance of the proportions to be used, and of other details, it may not be so easy to crystallise out the disodium salt, and here the very ready separation and purification of the trisodium salt becomes advantageous. From the trisodium salt the disodium imidosulphonate is prepared by treating the crystals with sulphuric acid slightly diluted until neutrality to litmus is reached. The solu-

tion with any crystalline precipitate (sulphate) is kept for some hours at or about 0° and the mother-liquor decanted, while still at that temperature, from the sodium sulphate. Evaporation, as already described, then yields the disodium salt in good crystals, pure or almost so.

Disodium imidosulphonate crystallises readily in large rhombic prisms. It is very soluble in water, slightly acid in reaction, and devoid of sulphurous taste. Its crystals, which contain water, do not effloresce, even in dry seasons, so that crystals of sodium sulphate among them may be easily recognised. In air kept dry by sulphuric acid, they slowly become opaque, but retain their shape, along with much of their lustre and hardness; while the rate of loss, even when they have been crushed, is so slow as to make a water estimation in this way almost impracticable. They can generally be kept for many weeks in bottles, and apparently for any time in a sufficiently dry atmosphere, but are liable to undergo hydrolysis into an acid mixture of sulphate and amidosulphonate. The solution of the salt is also unstable, but it may be heated for a short time to a moderate degree and yet escape change. The crystals can hardly be heated to 100° without some decomposition.

The decomposition of disodium imidosulphonate that almost unavoidably attends the process of removing the water of crystallisation, also complicates observation of the effects of a high temperature upon it. For, having been to some extent converted by this water into sulphate and amidosulphonate, it gives at first when heated only ammonia, which comes from the reconversion of the amidosulphonate into imidosulphonate. But at a higher temperature, this is followed by products the same as in the case of the dipotassium salt, namely, nitrogen with more than double its volume of sulphur dioxide, small brown and white sublimates, and sodium sulphate. The sodium salt

appears to be much more fusible than the potassium salt, but then in the state examined the latter was practically pure while the former was mixed with the products of its hydrolysis, which must have affected its fusibility.

Disodium imidosulphonate in aqueous solution gives, like the much less soluble dipotassium salt (Berglund), no precipitates with many of the usual metal salts. But it gives when in moderately concentrated solution a precipitate of dipotasssium imidosulphonate with solutions of potassium salts; of trisodium imidosulphonate with sodium hydroxide; and of a barium sodium imidosulphonate with barium hydroxide. Concentrated ammonia-solution also gives abundant precipitation. The precipitate, when the disodium imidosulphonate is unmixed with other salts, is a sodium ammonium salt (p. 75). But when a little sodium nitrate, chloride, or other sodium salt is added before the ammonia, the latter, in concentrated solution, precipitates pure trisodium imidosulphonate. The two forms of precipitation are expressed by the respective equations:—

$$5HN(SO_3Na)_2 + 5NH_3 = 2Na_5AmN_2(SO_3)_4 + AmN(SO_3Am)_2:$$

$$HN(SO_3Na)_2 + NH_3 + NaNO_3 = NaN(SO_3Na)_2 + AmNO_3.$$

The precipitation by ammonia is in both cases greatly increased by excess of the reagent, and with the disodium imidosulphonate added in powder to strong ammonia-water, much of the triammonium salt also precipitates.

Hydroxy-lead acetates, it was known, give copious precipitates. Normal lead acetate also gives scanty precipitation of a lead imidosulphonate. even when the acetate is truly normal as described, and the disodium salt in solution faintly acid to litmus (pp. 86, 88). Lead nitrate occasions no true precipitation, but, unlike the acetate, it rapidly effects hydrolysis and consequent precipitation of a little lead sulphate.

Both mercury nitrates are also precipitating agents, acting here as they do when added in excess to a solution of trisodium imidosulphonate (p. 101). Mercuric oxide, especially the precipitated form, reacts with it to give mercuric disodium imidosulphonate (p. 102). Warmed with a moderately concentrated solution of the salt, the mercuric oxide dissolves and, on cooling and standing, the solution deposits crystals of the double salt. Silver hydroxide is rapidly converted by solution of disodium imidosulphonate into the sparingly soluble argentum disodium salt (p. 95) Cupric hydroxide is without action.

Evaporated on the water-bath with sodium carbonate, or even with acetate, disodium imidosulphonate is partly converted into trisodium imidosulphonate, while carbon dioxide or acetic acid escapes. In the case of using acetate, therefore, there is presented the striking phenomenon of a strongly acid vapour rising from a well-marked alkaline liquor.

In analysing the salt, we hydrolysed it by heating it with hydrochloric acid to 150° in sealed tubes for some hours, here following Raschig in his analysis of the potassium salts. The composition of the salt is expressed by the formula—$HN(SO_3Na)_2.(OH_2)_2$, the results of analysis being—

	Calc.	Found.			
Sodium	17·90	17·77	17·76	—	—
Sulphur	24·90	—	—	24·88	24·78
Nitrogen	5·45	—	—	5·40	—

Trisodium imidosulphonate.—This salt is prepared from the disodium salt by adding sodium hydroxide to its strong solution. It is unnecessary to have the disodium salt pure and in crystals. After following the process for getting this salt so far as to separate one good crop of crystals of sodium sulphate, the liquor is diluted with

twice its volume, or more, of water, treated with sodium hydroxide solution till it tastes slightly caustic, and then cooled in an ice-box. The trisodium salt begins to separate almost at once after the addition of the sodium hydroxide, but the water previously added retards the separation and enables the crystals to grow in the cooled liquor sufficiently large to be afterwards better freed from their mother-liquor. Strained out, and pressed in calico, the salt becomes almost pure, and can be rendered fully so by recrystallisation.

Trisodium imidosulphonate is a very stable and easily prepared salt. It forms thin, overlapping, hexagonal plates, which may grow to considerable size, but are seldom if ever to be seen single and perfect, or with much thickness. The crystals readily effloresce in dry air, and have a mildly alkaline taste. The salt is sparingly soluble in very cold water and very soluble in hot water, taking for dissolution at $27\frac{1}{2}°$ about 5·4 parts of water. Its solution readily shows supersaturation. It is alkaline to litmus, and even to phenol-phthaleïn, but exercises no action whatever on iodine-solution, and is precipitated from its aqueous solution by alcohol without decomposition. It can be repeatedly recrystallised without any loss of alkali. Aqueous solutions are quite stable even when continuously boiled. The crystals melt when gently heated, and their water may be rapidly boiled off without causing any decomposition of the salt. In a vacuum over sulphuric acid they lose only eleven molecules of the twelve they contain. The salt loses more water when heated, but even at 160° it retains some, possibly through hydrolysis, so that heated until decomposition begins, it yields always a very little water and ammonia (gas and sublimate). Heated in a vacuum, however, it yields even still less, so that atmospheric moisture seems to be active in hydrolysing it when heated in air.

The presence of hydrogen in the dried salt might have particular

significance, since Fremy and, after him, Claus have represented the dipotassium salt to contain three atoms of hydrogen instead of the one given by the imide constitution and displaced by metal in the normal salts. But careful combustion of some grammes of the dried salt with copper oxide gave us in one case only 0·22 per cent., and in another only 0·24 per cent. of water, calculated from the hydrated crystals, and such a percentage corresponds to only about one-ninth of an atom, instead of the extra two, of hydrogen required by the sulphammonate constitution given by Claus.

When dried trisodium imidosulphonate is heated somewhat strongly in an open tube it yields, with fusion and effervescence, nitrogen and some sulphur dioxide, and sulphur which sublimes. The saline mass becomes also very dark-coloured from the presence in it of sulphur, and gradually solidifies, until at a commencing red-heat it forms a semi-fused *hepar sulphuris*. By heating in a vacuum, proof is obtained that the sulphur dioxide comes only from the action of the air upon the sulphur. From 350° to 440,° gas of only very low tension is given off, along with sulphur vapour forming a sublimate of drops. Just below the softening point of good soft lime-glass, the salt fuses and effervesces. The gas consists entirely of nitrogen. Some of the sulphur remains in the fused mass, partly free, partly as thiosulphate. A trace of the ammonia compound with sulphur dioxide forms a sublimate. Apart from this the reaction is expressed by the equation—

$$2NaN(SO_3Na)_2 = N_2 + S + 3SO_4Na_2$$

Acids readily dissolve trisodium imidosulphonate, yielding neutral solutions of the disodium salt when the quantity of acid is equivalent to one-third of the sodium. Remarkable is the action of concentrated sulphuric acid which, being not in excess, dissolves the crystals with marked fall of temperature, although when in quantity more than

enough to form the disodium salt, its addition causes heating. From the neutralised solution sulphate and disodium imidosulphonate can be crystallised out. Carbon dioxide also decomposes the trisodium salt in water, sodium acid-carbonate being precipitated, if the water present is not too great. Concentrated ammonia-water precipitates the trisodium salt free from ammonia, and with its usual water of crystallisation. Ammonium salts suffer double decomposition with the trisodium salt, ammonia then becoming free by the decomposing action of water upon some of the triammonium salt. When concentrated solutions of the trisodium salt and of a potassium salt, such as the nitrate, are mixed, nothing is observable, but on neutralising with an acid, there is precipitated the dipotassium salt in crystals.

Trisodium imidosulphonate, unlike the disodium salt, precipitates many metallic salts, giving in some cases, however, only hydroxides. Precipitates of imidosulphonates are obtained with silver nitrate, the mercury nitrates, lead salts, and barium salts, while with calcium chloride crystals may slowly form which are of characteristic appearance. Specially noteworthy is the fact that silver nitrate added in different proportions yields three precipitates distinct both in appearance and in composition. The compounds formed by these reagents are described later in this paper. Mercuric chloride has apparently no action upon trisodium imidosulphonate, for nothing separates, and the reaction with litmus remains alkaline. Mercuric oxide dissolves to a limited extent in solutions of the trisodium imidosulphonate, the more in quantity (in proportion to the trisodium salt) the weaker these solutions are, and makes them somewhat caustic-alkaline (see, in this connection, the reaction of mercuric sodium imidosulphonate with alkali hydroxide, p. 104).

It was not practicable to get the crystals of trisodium imidosulphonate exactly dry for analysis, because the thin plates adhering

together retained mother-liquor, and, on the other hand, during much crushing and pressing with paper they effloresced. Avoiding the latter source of error, our results were $12\frac{1}{4}$ OH_2, the $\frac{1}{4}$ representing one per cent. of moisture. The numerical details of the water determinations, which are a little complex, are the following :—

	Calculated.	Found.		
Moisture	1·00 }	43·66	43·56	Loss *in vacuo*
11 mols.	42·70 }			
1 mol.	3·88	3·71	3·64	Loss at 110°
		(0·23	0·27	Retd. at 110°)
Total	47·59	47·60	47·47	

We have already given (p. 72) the percentage of retained water, as actually found in the salt dried with special care, as 0·22 and 0·24 per cent. The above numbers, 0·23 and 0·27, are difference numbers.

The analysis of the dried salt gave—

	Calc.	Found.
Sodium	28·40	28·24
Sulphur	26·34	26·46
Nitrogen	5·76	5·92

The formula of the salt is therefore $NaN(SO_3Na)_2.(OH_2)(OH_2)_{11}$.

Sodium ammonium imidosulphonates.

Possibly a salt two-thirds sodium and one-third ammonium-imidosulphonate might be got in presence of an excess of the triammonium salt, but this is doubtful because of the sparing solubility of the following salt in presence of ammonia, and because ammonia and another sodium salt convert disodium to trisodium imidosulphonate (p. 69).

Pentasodium ammonium imidosulphonate.—From dilute solution this salt crystallises with seven molecules of water, $Na_5AmN_2(SO_3)_4,(OH_2)_7$, but from concentrated saline liquors it crystallises with less water. To prepare the fully hydrated salt, strong ammonia-water is added in large excess to a solution of disodium imidosulphonate, which must be free from sulphate or other salt. Separation of the salt begins at once or very soon and continues for some time if the liquor is kept nearly ice-cold. The salt forms minute prisms which bear moderate washing, with concentrated ammonia-water, and can be drained on a tile unchanged if under close cover. When dry, the salt does not smell noticeably of ammonia. The numbers obtained by calculation and experiment are as follows :—

	Calc.	Found.
Sodium	18·95	19·19
Ammonium	2·97	2·85
Sulphur	21·09	20·95

A salt with $2\frac{1}{2}H_2O$ only, falls as a crystalline precipitate on adding strong ammonia-water to a solution of trisodium imidosulphonate and its equivalent at least of ammonium nitrate (3 mols.). Being but very little soluble in ammonia-water, it can be properly washed with this, and may then be drained dry on a tile under cover without change. Heated it loses water and ammonia without suffering aqueous fusion. It contains no nitrate. Analysis gave—

	Calc.	Found.
Sodium	21·86	22·01
Ammonium	3·42	3·40
Sulphur	24·34	24·40

For notice of a monohydrogen sodium ammonium imidosulphonate, one in which the ammonium greatly predominates over the sodium, see p. 77.

Hydrogen sodium ammonium imidosulphonate nitrate.

On dissolving trisodium imidosulphonate and then its equivalent of ammonium nitrate (three molecules) in half their combined weight of hot water in a closed nearly full vessel, a cold solution is obtained, which sometimes remains unchanged, super-saturated, sometimes slowly deposits the trisodium salt again of the ordinary form, but in thicker crystals than usual. The attempt to redissolve the crystals by dipping the vessel in hot water readily succeeds but is generally attended with the copious separation of some white opaque salt (sodium ammonium imidosulphonate?). On the vessel being left for some days in a cold place, with occasional agitation, the trisodium salt re-forms in crystals, and the opaque salt redissolves. When proceeding differently, the fresh mixed solution of the trisodium salt and ammonium nitrate in half their weight of water is placed on the steam bath in an open glass capsule, ammonia freely escapes, and soon groups of small prisms form, evidently in consequence of the loss of ammonia by the solution and not its mere evaporation, for the addition of some water has little effect upon the quantity of these crystals. The crystals could not be separated from their highly concentrated mother-liquor in a state fit for trustworthy analysis, but we are pretty confident of their being the salt now to be described.

A well-crystallised double salt of definite composition is obtainable by evaporating a solution of trisodium imidosulphonate with at least its equivalent of ammonium nitrate, until the liquor is nearly neutral, then adding water to redissolve any crystals which have formed, and leaving the product to cool slowly. Throughout, care must be taken that the liquor does not become acid during evaporation, by adding if necessary a drop or two of ammonia. Small flat, thick prisms are

obtained which are anhydrous and stable in the air. They cannot be washed, and must be freed from their mother-liquor by pressure in paper. The new salt is of a composition which may be represented as being that of one molecule of diammonium imidosulphonate with one of sodium nitrate. Prepared from a good excess of ammonium nitrate the crystals proved to be almost pure, but holding a little water (a); with the use of less ammonium nitrate, the salt showed on analysis the presence of a very slight excess of disodium imidosulphonate. The yield of salt was three-fourths of the possible quantity. The mother-liquor on evaporation gave crystals of sodium nitrate and some more of the salt. The double salt cannot be recrystallised, its solution yielding, first, good crystals of a salt which may be described as (anhydrous) diammonium imidosulphonate containing 3 per cent. of sodium, and, after this salt, crystals of the double salt. Since the mother-liquor of the double salt yields sodium nitrate, it would seem that double decomposition occurs between sodium imidosulphonate and ammonium nitrate.

The presence of nitrate in the double salt cannot be directly indicated in the ordinary way, but may be at once detected by a new reaction of delicacy and of some general interest. The salt freely effervesces when immersed in strong sulphuric acid, the gas being nitrous oxide with a little nitrogen. No nitrosyl sulphate is produced. Quite a minute quantity of nitrate mixed or combined with an imidosulphonate is shown by dropping the dry solid into a little of the acid, and not mixing them up too much. The reaction needs careful study. We ventured to determine the nitric acid in the salt by means of this reaction, taking the gas as nitrous oxide only, and got a good result, but do not feel justified in entering it in the tabular statement below

In one preparation we determined the nitric acid; in the other it

was undetermined, except in the experiment above referred to. The nitric acid was estimated by evaporating the salt with barium hydroxide to expel ammonia, filtering off the barium imidosulphonate, removing barium as carbonate, and then putting the concentrated solution of sodium nitrate into the nitrometer. The result was low, but that is hardly to be wondered at. Water, in that preparation in which it was present in any quantity, was estimated by heating the salt in dried air, and the result can be only approximate, for slight hydrolysis, fixing water, is hardly to be avoided.

	Calc.	a.	b.
Sodium	7·77	8·00	8·42
Ammonium	12·16	11·71	11·37
Sulphur	21·62	21·62	22·08
Nitric acid, NO_3H.	21·28	19·93	—
Water	0·00		

The preparation *a* was formed in presence of good excess of ammonium nitrate, and *b* in presence of little more than enough of it. The calculation is for $HN(SO_3Am)_2.NaNO_3$.

Hydrogen sodium potassium imidosulphonate nitrate.

The very sparingly soluble dipotassium imidosulphonate is hardly affected by digestion with a cold dilute solution of sodium nitrate, and when dissolved in it by heating crystallises out again almost unchanged, but when left in a concentrated solution of sodium nitrate it removes this salt from it. Taking about equivalent quantities of the salts, $HN(SO_3K)_2 : 3NaNO_3$, and leaving the solution of the nitrate standing over the dipotassium imidosulphonate in a loosely covered beaker for some days, the latter gives place to a caked mass of crystalline granules, appearing under the microscope homogeneous but with

no well-defined forms. The decanted mother-liquor evaporated a little gives more of the granules, but no crystals of sodium or potassium nitrate. The granular mass drained and pressed on a tile has a composition closely approaching that of one molecule of dipotassium imidosulphonate to one of nitrate, the nitrate being of potassium and sodium in single atomic proportions, $NaNO_3,KNO_3,2HN(SO_3K)_2$. It is decomposed by water, and is anhydrous.

	Calc.	Found.
Sodium	3·32	3·61
Potassium	28·23	27·34
Sulphur	18·48	18·89
Hydrogen nitrate	18·20	18·01

The two potassium salts or the two sodium salts do not form double salts together. Disodium imidosulphonate and potassium nitrate suffer double decomposition. The formula of the salt just described shows also that, to some extent, this double decomposition is reversible.

In our next communication we shall have to describe still more remarkable combinations of nitrates with oximidosulphonates.

Barium imidosulphonates, simple and double.

Barium imidosulphonate.—This salt has been prepared by Berglund, but no particulars of it have been published, except possibly in Swedish. It is obtained as a voluminous, coherent precipitate when trisodium imidosulphonate is added gradually to good excess of barium chloride. Drained on the porous tile, after thorough washing, it is still very bulky, and hangs together in soft flocks, more like some organic salt than an inorganic barium salt. It retains moisture in the interstices of its matted texture with remarkable obstinacy, so

that it can not be drained dry on the tile, or rendered dry in the desiccator except at the surface of its masses. Yet it is not in the least deliquescent, and becomes dry in the air when spread out in very thin layers; indeed in very dry air it is even slightly efflorescent. Analysis shows it to contain water, but it loses this only very slowly even at 115.° Under the microscope it is seen to consist of interlacing, very long, slender needles. It is only very slightly soluble in water, only sufficiently so to cause the water to give a very slight opalescence with sulphuric acid. It is alkaline to litmus. It is decomposed a little by solutions of alkali carbonates. It may be dissolved in dilute nitric acid, and, with very rapid treatment, reprecipitated unchanged by barium hydroxide. This is a property which, with precautions, may be taken advantage of to remove sodium salt, which is nearly always present in the original precipitate, but always absent after second precipitation. Only once did we get the original precipitate free from sodium. Analysis gave—

	Calc.	Found.
Barium	48·38	47·91
Sulphur	15·08	15·04
Nitrogen	3·29	3·31

The calculation is for $Ba_3N_2(SO_3)_4, (OH_2)_5$. The alkalinity of the salt, measured by decinormal acid and methyl orange, proved equal to one-third of the barium.

Ammonia added to a solution of the salt next described converts it into tribarium imidosulphonate insoluble, and ammonium imidosulphonate.

Barium hydrogen imidosulphonate.—This salt has been briefly noticed by Jacquelain and by Berglund. We have prepared it from the tribarium salt, thoroughly free from sodium, by cautious treatment with dilute sulphuric acid equivalent to slightly less than one-third

of the barium, filtering off barium sulphate, and evaporating in a desiccator. Minute, brilliant, orthorhombic crystals are obtained which are moderately soluble in water. The salt is stable in the desiccator, or in the open air in dry weather, but soon hydrolyses in a bottle. In dried air it is stable even when heated considerably, but it hydrolyses in ordinary air at and above 140°. Rapidly heated it decomposes suddenly into a cloud of barium sulphate. Its solution is acid to litmus but neutral to methyl orange.

Proceeding in almost the same way as that followed by us, Woronin obtained apparently the same salt, but he gave to its composition a molecule of ammonia more, calling it *barium sulphamate*. Since our preparation contains a molecule of water, the determination of barium and of sulphur agrees with both formulæ—$Ba(H_2NSO_3)_2$ 'sulphamate,' and $BaHN(SO_3)_2, OH_2$ imidosulphonate.

Quantitative analysis and the properties of the salt agree with the formula, $BaHN(SO_3)_2, OH_2$

	Calc.	Found.	
Barium	41·51	41·67	41·65
Sulphur	19·39	19·51	—

In warm solution barium hydrogen imidosulphonate reacts with mercuric oxide to form the double salt.

Barium ammonium imidosulphonate.—By adding one molecule of barium hydroxide to one of triammonium imidosulphonate ('neutral ammonium sulphamate') Woronin observed the formation of a 'soluble' barium salt, with the evolution of ammonia. He made no statement as to its composition. On adding barium hydroxide gradually to triammonium imidosulphonate until it nearly ceased to give a precipitate we obtained a very slightly soluble salt as a precipitate which was a barium ammonium imidosulphonate. It was not fully analysed, but gave a percentage of barium of 41·87 while

calculation for $BaAmN(SO_3)_2$ requires 41·64 %. But it may, not improbably from what follows, have been a hydrous salt with less ammonia.

The experiment repeated with diammonium imidosulphonate gave us a precipitate at first gelatinous but becoming granular on standing. Under the microscope it was seen to consist of granular crystals intermixed with a very few slender needles of the tribarium imidosulphonate insignificant in amount. Its alkalinity determined by decinormal acid and methyl orange showed it to be a normal salt, the alkalinity being equal to one atom of univalent base to two atoms of sulphur present. After hydrolysis the total ammonia, as well as the sulphuric acid, was estimated. The results agree well with the calculation for $Ba_5Am_2N_4(SO_3)_8(OH_2)_8$, a five-sixths barium salt, corresponding, therefore, to the sodium ammonium salt :—

	Calc.	Found.
Barium	43·88	43·84
Sulphur	16·40	16·53
Nitrogen	5·38	5·27

It is here assumed that the observed presence of a very little tribarium salt could be disregarded. The assumption that the preparation was a mixture $Ba_3N_2(SO_3)_4, (OH_2)_5 + 2[BaAmN(SO_3)_2, (OH_2)_{1.5}]$ could not possibly be admitted as in accordance with the microscopic appearance of the preparation.

Jacquelain obtained, by adding diammonium imidosulphonate to a slight excess of baryta water, a precipitate which he very fully analysed, and found to be composed in accordance with the empirical formula, $(NH_3)_2(BaO)_2(SO_3)_3$. Instead of this impossible formula we venture to write $Ba_4AmN_3(SO_3)_6, (OH_2)_3$, the formula of an eight-ninths barium ammonium imidosulphonate, since the calculated numbers agree still better with his analysis than those for his formula

	Calc.	Found. (Jacquelain.)
Barium	47·99	48·26
Sulphur	16·81	16·81
Nitrogen	4·90	4·97
Hydrogen	0·88	0·88

Using ammoniated solution of barium chloride instead of baryta water, he got nearly the same results.

Barium potassium imidosulphonate.—This is a nearly insoluble, crystalline salt, preparable by heating dipotassium hydrogen imidosulphonate with baryta water. It has been described by Berglund, and was known to Fremy. We have not analysed it, and believe it has not been analysed by others.

Barium sodium imidosulphonate.—Barium chloride, in dilute solution, very slowly added with constant stirring to an excess of trisodium imidosulphonate gives a crystalline precipitate, which under the microscope is not seen to contain any tribarium imidosulphonate in admixture. It is alkaline to litmus, very sparingly soluble in water, readily soluble in nitric or hydrochloric acid, and largely but never completely decomposed by ammonium carbonate. It contains water, but does not noticeably lose weight even at 120.° Heated quickly to a higher temperature, it is dissipated as a cloud of barium sulphate and gases. Five preparations made at different times show close agreement in composition, with the exception of that first prepared (which is entered in our note-book as not pure). The numbers agree with those calculated for $Ba_{11}Na_8N_{10}(SO_3)_{20}, (OH_2)_{13}$, which may be written as $8BaNaN(SO_3)_2, OH_2 + Ba_3N_2(SO_3)_4, (OH_2)_5$:

	Calc.	*a*	*b*	*c*	*d*	*e*
Barium	41·12	42·35	40·05	41·17	40·45–41·07	40·85–40·91
Sodium	5·02	5·08	5·26	4·93	—	5·25
Sulphur	17·46	18·80	—	—	17·54	17·56

The barium and sodium were determined by cautious ignition of the salt alone, then with sulphuric acid, and boiling out with water. The residue was weighed as barium sulphate, and the soluble matter after re-ignition weighed as sodium sulphate. The sodium of *d* was lost. The sulphur was estimated by hydrolysis followed by precipitation with barium chloride, but the barium sulphate was here weighed in two quantities, that formed by hydrolysis and that by the barium chloride.

Calcium imidosulphonates, simple and double.

Normal calcium imidosulphonate, $Ca_3N_2(SO_3)_4$, is not formed by reaction between sodium or ammonium imidosulphonate and calcium chloride. When a solution of diammonium imidosulphonate is treated with pure, soft, calcium hydroxide in the calculated quantity, ammonia is at once liberated and is all expelled by two or three evaporations of the solution to dryness. The salt left dissolves and forms concentrated solutions crystallising in transparent prisms. But although the salt dissolves at first freely, the last portions are more difficult to bring into solution, and it may be that water partly decomposes the salt into the calcium hydrogen salt and calcium hydroxide. The salt was not analysed, except that a calcium determination was made, according to which the crystals would have ten molecules of water to two atoms of nitrogen in the salt.

Calcium hydrogen imidosulphonate is obtained in radiating groups of fine needles when diammonium imidosulphonate and the calculated quantity of calcium hydroxide are mixed, and the solution repeatedly evaporated to expel all ammonia. The calcium hydroxide dissolves quickly after it is added. The salt is soluble and has not been analysed.

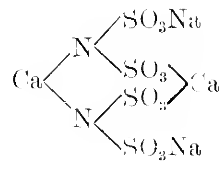
Calcium ammonium imidosulphonate is a sparingly soluble salt obtained when the calculated quantity of calcium hydroxide is dissolved

in diammonium imidosulphonate solution, this quantity being in the proportion, $Ca(OH)_2 : Am_2HN(SO_3)_2$. The salt rapidly crystallises after the dissolution of the lime. It has not been analysed.

Calcium sodium imidosulphonate.—To a hot concentrated solution of trisodium imidosulphonate about the calculated quantity of calcium chloride solution to form the tricalcium salt is added, and the mixture left to cool. Not the tricalcium salt but the calcium sodium salt crystallises out in groups of hard brilliant prisms. The salt can be recrystallised, and thus purified if necessary. It is sparingly soluble in cold water, and reacts alkaline. It only very slowly takes carbonic acid from the atmosphere. Analysis corresponds to the composition $CaNaN(SO_3)_2, (OH_2)_3$—

	Calc.		Found.	
Calcium	13·75	13·95	14·07	—
Sodium	7·90	7·72	—	—
Sulphur	21·99	—	22·21	—
Nitrogen	4·81	—	—	4·88

The relative positions of the calcium and sodium are not very certain. For sake of symmetry the sodium might be put in the imide relation, but such a procedure has nothing to support it. Taking the sodium to act as the more basylous element, the formula becomes—



Lead imidosulphonates.

A lead imidosulphonate was used by Jacquelain, and again by Berglund, to prepare imidosulphonic acid; but we know of no publication of particulars concerning the salt employed.

Disodium or other bibasic imidosulphonate does not precipitate with lead nitrate and only feebly with normal lead acetate. It is therefore necessary to take at least either the trisodium imidosulphonate, or else a basic lead acetate, to get an insoluble salt in sufficient quantity. This salt is always a basic or hydroxy salt, so that its separation renders its mother-liquor much less basic than the mixed solutions employed. Hence the common retention of lead salt and imidosulphonate together in the mother-liquors, and also the fact that either one or the other must be present in them. In fact, only by the use of a basic lead salt is it possible to throw all imidosulphonate out of solution, and only by using the trisodium salt is it possible to precipitate all lead. Oxy-lead acetate and lead nitrate or normal acetate precipitate different salts with trisodium imidosulphonate, but all the metal of either salt precipitated is lead. One other point to be here noticed as affecting precipitation, is the solubility in solution of normal lead acetate of the lead imidosulphonates insoluble in water. The solution is alkaline and freely precipitable by carbonic acid, and may be regarded as containing basic lead acetate and the soluble lead hydrogen imidosulphonate. Although a superficial examination of the reactions of imidosulphonates with lead salts may suggest the view of their being ill-defined, fuller investigation proves them to be quite definite.

The *normal salt*, $Pb_3N_2(SO_3)_4$, appears not to exist. The first addition of lead acetate or nitrate precipitates a basic salt with trisodium imidosulphonate; and basic lead imidosulphonate treated with small quantities of nitric acid dissolves as a whole to yield a neutral or slightly acid solution, the undissolved part remaining unchanged (p. 89).

The *lead hydrogen salt*, $HN(SO_3)_2Pb$, appears to exist in solution, but the attempt to separate it leads to decomposition. Its solution can

be prepared by treating either basic salt with somewhat less than enough dilute sulphuric acid, and decanting the clear solution from the sulphate and undecomposed basic salt. The solution can be preserved unchanged in absence of acetates for a very short time only. It has a slight acid reaction with litmus. Left in a desiccator it suffers decomposition into amidosulphonic acid and lead sulphate, slowly at first, more rapidly as the solution grows concentrated. Heating the solution effects the same change. Alcohol gives a voluminous precipitate gradually giving place to a crystalline deposit partly adhering to the sides of the vessel, nearly insoluble in water, and apparently sulphate.

Hemihydroxy-lead imidosulphonate, $HOPbN(SO_3PbOH)_2$.—Lead nitrate and trisodium imidosulphonate in solution brought together in widely varying proportions yield this salt as a precipitate, voluminous at first, but soon becoming dense and granular. Disodium imidosulphonate and sodium nitrate are the other products of the reaction :—

$$3Pb(NO_3)_2 + 3OH_2 + 4Na_3N(SO_3)_2 = (HOPb)_3N(SO_3)_2 + 3HNa_2N(SO_3)_2 + 6NaNO_3$$

The precipitation is closely quantitative. With the trisodium salt not used in excess the mother-liquors are neutral in reaction with methyl orange.

Normal lead acetate may be used in place of the nitrate, but only with less perfect results. The best way to proceed is to mix the solutions rapidly together in something approaching the right proportions ; or the two salts may be rubbed together in the solid state and only then treated with water. In either of these ways, when the proportion of either salt is not very many times greater than it should be, a product of constant composition is obtained, which differs however from the pure lead salt in having about one-seventy-third of

the lead replaced by hydrogen. The weight of the precipitate obtained from a given quantity of trisodium salt, is greater with lead acetate than with the nitrate, and greater as the excess of acetate used is greater. The acetate mother-liquors, though alkaline to methyl orange, are acid to phenol-phthaleïn, and after replacing the lead in them by sodium, by precipitating with normal sodium oxalate, can be titrated by sodium hydroxide and phenol-phthaleïn as indicator. Thus tested these liquors proved to contain free acetic acid, and in increasing quantity with that of the acetate used, and, therefore, of the precipitate produced. In these experiments particularly excellent *normal* lead acetate was used. It appeared from the experiments made that about one-eighth of the excess of lead acetate used must react with disodium imidosulphonate, a product of the main reaction, as already pointed out, the change being as follows:—

$$3Pb(O\overline{Ac})_2+3OH_2+HNa_2N(SO_3)_2=(HOPb)_3N(SO_3)_2+2NaO\overline{Ac}+4HO\overline{Ac}$$

The acetic acid thus generated, acting so as to reverse the reaction, is probably the cause of the very slight replacement of lead by hydrogen in the salt precipitated by acetate.

When the trisodium imidosulphonate solution is added by degrees to the lead acetate, a gelatinous precipitate like lead hydroxide forms and then redissolves in the lead acetate. When enough of the sodium salt has been added to cause a permanent precipitate this is somewhat slimy and only slowly becomes pulverulent, while the walls of the vessel get coated with crystalline precipitate. The mixed product is unfit for analysis. When the lead acetate is added gradually to the sodium imidosulphonate, the precipitate also remains voluminous and somewhat gelatinous, and is probably not a pure product. When after adding not too much sodium salt, the liquor is quickly filtered, it slowly yields a few brilliant crystals of what is

evidently, both from its appearance and an imperfect analysis, the hydroxy-lead salt.

The hydroxy-lead salt in perfect brilliant microscopic prisms can be obtained very pure by treating the more basic lead salt next described, with little more nitric acid than that calculated to remove the excess of lead. The nitric acid can be clearly seen to dissolve a portion of more basic salt, and almost at once to deposit a crystalline precipitate and incrustation of the hemi-basic salt. If instead of stopping the addition of nitric acid when enough has been used, more is added, the crystalline salt proportionately dissolves without reprecipitating. The nitric solution is neutral to methyl orange, so that the reaction is, in the dissolution of the more basic salt,—

$$(HOPb)_3N(SO_3)_2, Pb(OH)_2 + 6HNO_3 = HPbN(SO_3)_2 + 5OH_2 + 3Pb(NO_3)_2$$

and then, this solution reacting with a further quantity of the more basic salt:—

$$2(HOPb)_3N(SO_3)_2, Pb(OH)_2 + HPbN(SO_3)_2 = 3(HOPb)_3N(SO_3)_2 + OH_2$$

This, in a manner, can be shown by pouring the nitric solution into a large excess of hydroxy-lead acetate solution, when the hemi-hydroxy-lead imidosulphonate also precipitates in the pure state. There is also the fact observed that the nitric liquor when soon poured off from undissolved more-basic salt contains much lead, whereas when it is left over the undissolved more-basic salt it goes on depositing a crystalline precipitate of the hemi-basic salt.

The modification of this process expressed by the equation—

$$3(HOPb)_3N(SO_3)_2, Pb(OH)_2 + HN(SO_3Na)_2 + 2HNO_3 = 4(HOPb)_3N(SO_3)_2 + 2NaNO_3$$

has also been quite successful, by adding the nitric acid very slowly and shaking well.

The hemi-hydroxy-lead imidosulphonate is an anhydrous salt, not counting its hydroxyl, very permanent, losing nothing at 100°,

insoluble in water, and scarcely, if at all, alkaline in reaction when in contact with wet litmus paper.

A number of preparations by different methods have been analysed with closely concordant results:—

	Calc.	Found.
Lead	73·40	73·21–73·35
Sulphur	7·56	7·50

The lead acetate precipitates gave from 72·31 to 72·78 °/₀ lead, and from 7·52 to 7·65 °/₀ sulphur. The sodium was determined in some of these preparations, and was found to be no more than 0·075 °/₀.

The pure salt was also titrated with volumetric nitric acid and its basicity found to agree with calculation. One gramme of salt took 48·71 ccs. of acid, instead of 47·28 ccs. calculated to just dissolve it and furnish a solution neutral to methyl orange.

Five-eighths-oxy-lead imidosulphonate, $(HOPb)_3N(SO_3)_2$, PbO or $Pb(OH)_2$.—This salt is precipitated on adding trisodium imidosulphonate to excess of basic lead acetate. The quantity of basic lead acetate which is needed is very large. Trial taught us to use not less than six molecules of hemi-hydroxy-lead acetate to one of trisodium imidosulphonate. The calculated quantity is five molecules, but an excess helps to keep sodium out of the salt. The reaction is as follows:—

$$Na_3N(SO_3)_2 + 5HOPbOAc = (HOPb)_3N(SO_3)_2{,}Pb(OH)_2 + 3NaO\overline{Ac} + Pb(O\overline{Ac})_2$$

and therefore it would be better probably to use a more basic lead acetate. Using as we did, about six molecules of hemi-hydroxy-lead acetate, the mother-liquor of the precipitate proved to be markedly basic still. Both precipitate and mother-liquor were quite free from sulphate. A very little sodium, but no acetic acid, was found in the precipitate, after it had been well-washed. It is readily soluble in

dilute nitric acid, insoluble in water. Its partial dissolution in nitric acid leads to the formation of the hemi-hydroxy-lead imidosulphonate. It has scarcely any action on moist red litmus with which it is left in contact. It is insoluble in solutions of its mother-salts, and unaffected in composition by them.

Heated even at 130° it loses no water. When sufficiently hot it gives off, first water, then ammonia, then again water, and then it blackens through formation of sulphide, and then it evolves sulphur dioxide. The loss of water and ammonia is explained by the equation :—

$$3(HO)_5Pb_4N(SO_3)_2 = 6PbO + 3PbSO_3 + 3PbSO_4 + NH_3 + N_2 + 6OH$$

Then the lead sulphite becomes, as usual by the rising temperature, partly lead sulphide and sulphate, partly lead oxide and sulphur dioxide. At a fusing heat the sulphide and oxide would of course react.

Two preparations were analysed, *a* and *b* :—

	Calculation.	Found.		
	$(HOPb)_3N(SO_3)_2$, PbO, $\frac{1}{2}H_2O$	*a*	*b*	*b*
Lead	76·81	76·79	76·05	76·82
Sulphur	5·94	5·82	6·00	—
Sodium	—	0·15	0·15	—

Behaviour of imidosulphonates as compound amines to other bodies.

We collect together here some facts already recorded in the paper which appear to show that imidosulphonates form compounds in which the nitrogen becomes quinquevalent as in ammonium salts.

First, in order of observation, there is the retention of one molecule of water at 110° by the trisodium salt. The potassium salt which crystallises with one molecule of water (Raschig) has not been examined in this regard. Next, there is the union of diammonium salt with sodium nitrate to form a well-crystallised compound which

may be formulated as $\begin{matrix} H \\ (NO_3) \end{matrix}\!\!>\!N\!<\!\begin{matrix} (SO_3Am)_2 \\ Na \end{matrix}$, and the similar compound of dipotassium imidosulphonate. Lastly, there is the more-basic lead salt described in the last part of the previous section of this paper. This may be written $\begin{matrix} Pb \\ | \\ O \end{matrix}\!\!>\!N\!<\!\begin{matrix} (SO_3PbOH)_2 \\ PbOH \end{matrix}$ or $(HO)N\!<\!\begin{matrix} (SO_3PbOH)_2 \\ (PbOH)_2 \end{matrix}$, if we, allowably by the analysis, reject or admit the presence of one molecule of water.

Silver imidosulphonates, single and double.

Silver nitrate and trisodium imidosulphonate will yield three different compounds, in which one, two, and three atoms of sodium are replaced by silver. Such an unusual behaviour in the salt of a polybasic acid gives to imidosulphonic acid a special interest.

The addition of silver nitrate in limited quantity causes a white precipitate which redissolves to be rapidly followed, unless the solutions are very dilute, by the separation of another compound. This consists of interlacing fibrous crystals filling the liquid as a felted mass, even when present in only small quantity. Thus, the whole may be well stirred and yet afterwards prove able to support the stirrer away from the sides of the beaker, though the actual volume of the salt, separated from its liquor and pressed, may be a small fraction of the whole.

Further additions of silver nitrate to the liquor and crystals cause generally more of the first white precipitate, and after that, when in sufficient quantity, the disappearance of both the precipitates together. The new solution becomes almost immediately full of a third precipitate, this time of very dense, hard, minute, separate crystals, which rapidly subside into a deposit that in appearance, in grating under the stirrer, and in movement when disturbed, most

closely resembles white sea-sand (silver-sand). A salt of the same composition as this often occurs as a hard, crystalline incrustation on the sides and bottom of the beaker, when to the mother-liquor of the felted crystals a little silver nitrate is added.

(Some experiments with potassium imidosulphonate have given us salts apparently analogous to those got from the sodium salt, but we have not further examined them).

*Triargentum imidosulphonate.**—By adding the sodium salt in limited quantity to silver nitrate, the white precipitate is also formed, and is now permanent in its mother-liquor. This precipitate, which is triargentum imidosulphonate, is very little soluble in water, and after long washing on the filter yields a milky filtrate. It is a relatively voluminous precipitate forming chalk-like masses when dry. It is without water of crystallisation, as was to be anticipated, but even when kept long at 110° retains 0·55 per cent. of water (possibly fixed by hydrolysis from the atmosphere), as determined by combustion with copper oxide. It bears heat well, but at a comparatively high temperature rapidly decomposes with or without fusion, giving in the open tube first a very little ammonia gas and sublimate, then sulphur dioxide, nitrogen, silver, silver sulphide, and silver sulphate. When heated in a vacuum, scarcely any sublimate is formed, and no change takes place till the temperature reaches 440°, when gas comes off, and blackening without fusion occurs. The residue consists of silver sulphate, silver sulphide, and silver, and the gas of nitrogen and sulphur dioxide. The proportions of these products vary a little, and apparently according to the mode of heating, a higher temperature giving more silver and sulphur dioxide. The first moiety of gas was thus found to consist of four volumes of nitrogen to five of sulphur dioxide, and the second moiety of two volumes of nitrogen to three of

* May possibly have been already described by Berglund in the *Lunds univ. Arsk.* 12.

sulphur dioxide, so that the decomposition lies between those expressed by the equations :—

$$4AgN(SO_3Ag)_2 = 2N_2 + 2SO_2 + 5SO_4Ag_2 + SAg_2$$
$$4AgN(SO_3Ag)_2 = 2N_2 + 4SO_2 + 4SO_4Ag_2 + 2Ag_2,$$

but nearer the first one.

The salt was prepared for analysis by adding the sodium salt gradually, with stirring, to excess of silver nitrate. In spite of the strongly alkaline reaction of the former, the mother-liquor of the precipitate proved to be neutral, as it should be, while the precipitate was not in the slightest brown, but brilliantly white. Being undecomposed by water it was washed. Its analysis gave, besides the 0·55 per cent. water mentioned above, the following results :—

	$AgN(SO_3Ag)_2$		Found.	
Silver	65·06	64·92	64·53	64·58
Sulphur	12·85	—	—	12·94
Nitrogen	2·81	—	—	3·16

Diargentum sodium imidosulphonate.—The sandy precipitate, described above, is a salt with two atoms of silver to one of sodium. It is sparingly soluble in water and very slightly decomposed by it, silver hydroxide being one of the products. It has a faint buff colour. Its crystals are seen under the microscope to be hexagonal plates, always single, which rotate polarised light. This salt, though otherwise anhydrous, retains a little water with the same firmness as that shown by the wholly silver salt. Its behaviour when heated closely resembles that of the latter, except that, naturally, sodium sulphate forms part of the residue, as well as silver sulphate.

To prepare the salt for analysis, the sodium salt was added gradually to a little less than two-thirds of its equivalent of silver nitrate. The sandy precipitate obtained was only slightly washed,

and then drained and dried on a tile. An incrustation (washed *in situ*) was also analysed. The results were :—

	$Ag_2NaN(SO_3)_2$	Incrust.	Precip.
Sodium	5·57	5·61	5·67
Silver	52·30	52·02	51·87
Sulphur	15·50	—	15·60
Nitrogen	3·39	—	3·45

Argentum disodium imidosulphonate.—This salt, which occurs as crystalline fibres, is the most soluble of the three salts, but is still only sparingly so. It is gradually and visibly decomposed by water, the change consisting apparently, partly in the formation of silver hydroxide and hydrogen disodium imidosulphonate, partly in resolution into trisodium salt and the diargentum sodium salt. Although readily produced it is, therefore, difficult to get pure. It retains a very little water, as usual, and behaves when heated almost like the preceding salt. For analysis it was pressed from its mother-liquor on a calico filter, and then between porous tiles. In preparing it the sodium salt was added to a little less than one-third of its equivalent of silver nitrate, and the mixture agitated in order to redissolve the white precipitate of silver salt first thrown down before the fibrous crystals began to form. Analysis gave—

	$AgNa_2N(SO_3)_2$	*a*	*b*	*c*	
Sodium	14·02	14·75	—	—	13·81
Silver	32·93	33·07	33·49	33·50	33·49
Sulphur	19·51		19·18	19·52	—
Nitrogen	4·27	—	—	4·29	—

Double mercury imidosulphonates.

Single mercury imidosulphonates seem to be incapable of existence.

Mercury potassium imidosulphonates.—Berglund prepared mercury dipotassium imidosulphonate, $HgN_2(SO_3K)_4$, from tripotassium imidosulphonate. We have prepared it from dipotassium imidosulphonate by adding mercuric oxide to it and heating the mixture with water, filtering hot if necessary, and leaving the solution to cool. The salt, which is nearly insoluble in cold water, separates out in very minute needles, sometimes arranged loosely, sometimes in spherical tufts, and is when dry of silvery lustre.

An *oxymercuric potassium salt*, analogous to the sodium salt (p. 105), seems to be among the products of the reaction between mercuric nitrate and potassium imidosulphonates, but we have not made any study of it.

The behaviour of mercury dipotassium imidosulphonate with acids, as observed by us, is of interest in connection with the theory of the constitution of these salts, and we shall refer to it later on when discussing that point (p. 110). When the salt, in a state of paste with water, is treated with nitric acid, it is converted into the very slightly soluble dipotassium imidosulphonate, all its mercury dissolving as mercuric nitrate. Washed and drained on a tile, the salt left undissolved has been found to be free of mercury and of sulphate. The same experiment can be carried out by the use of dilute sulphuric acid, but less satisfactorily. The nitric acid used may be strong and in excess, but the sulphuric acid must be dilute and not in excess, and is apt to give a small precipitate of mercuric salt. Hydrochloric acid exchanges hydrogen for both metals, and effects extremely rapid hydrolysis.

Mercury barium, $HgN_2(SO_3)_4Ba_2$, *and other double mercury imidosulphonates.*—Mercury dipotassium imidosulphonate gives, according to Berglund, a series of double salts in which the potassium has been replaced by another base. It follows that these salts can also be

prepared from the mercury sodium salts, but we have only made ourselves familiar with the mercury barium salt, particularly noticed by Berglund. It is a lustrous, crystalline, dense salt, almost insoluble in cold water.

Mercury dihydrogen imidosulphonate.—When mercury barium imidosulphonate is treated with dilute sulphuric acid, not in excess of the barium, it appears to be sharply converted into barium sulphate and mercury hydrogen imidosulphonate. This observation made by Berglund we have confirmed and extended. We worked quantitatively and were careful to use the sulphuric acid in slight deficit, and to filter quickly, very easy to do in consequence of the dense state of the barium sulphate. But already hydrolysis of the imidosulphonate into sulphate and amidosulphonic acid had begun when we tested the filtrate. It progressed steadily, so that in ten minutes the sulphuric acid had grown to a considerable quantity, showing that mercury hydrogen imidosulphonate is stable to a less degree even than simple hydrogen imidosulphonate. In about ten minutes also, the solution began to grow turbid, and let fall an oxymercuric salt, probably amidosulphonate. The fresh solution when almost neutralised with potassium hydroxide gave a slight white turbidity, doubtless of oxymercuric potassium imidosulphonate, which dissolved when a little more alkali was added. The slightly alkaline solution, thus prepared, soon began to deposit minute crystals which we fully identified as mercury dipotassium imidosulphonate. Thus, we had proved, so far as might be done, that the fresh filtrate from the barium sulphate is really a solution of mercury hydrogen imidosulphonate.

Oxymercuric hydrogen imidosulphonate.—A second salt of mercury and hydrogen can be readily obtained, which is at once basic and acid salt, as its name indicates. Highly concentrated mercuric nitrate solutions, even when free as possible from nitric acid, dissolve, to clear

and comparatively stable liquors, solid potassium or sodium imidosulphonates. When sufficiently heated these liquors allow the imidosulphonate to hydrolyse into amidosulphonate, but when, without heating, they are diluted, they deposit oxymercuric hydrogen imidosulphonate. The same salt is almost immediately precipitated when trisodium imidosulphonate in solution is added to mercuric nitrate.

To prepare this salt take about five parts by weight of mercuric nitrate solution, undiluted and equal to about half its weight of mercuric oxide, and pour into it with stirring a cold solution (necessarily dilute) of one part of trisodium imidosulphonate. Precipitation of th- salt begins almost at once and is finished in a few minutes. The mother-liquor contains still much mercuric nitrate and some imidosulphonate, besides sodium nitrate and much nitric acid. Addition of a little more trisodium salt causes scarcely any more precipitation, while that of a large quantity throws down the oxymercuric sodium salt. The brilliant white and voluminous oxymercuric hydrogen salt hydrolyses only very slowly in its acid mother-liquor, because of the inhibitory action of the mercuric nitrate. It is to be washed repeatedly by subsidence and decantation with abundance of cold water and drained till dry on a tile or filter. Where less mercuric nitrate has been taken, and the precipitate contains sodium (as oxymercuric sodium salt), either digestion for a day with concentrated mercuric nitrate solution and washing. or else, without use of mercuric nitrate, continued and thorough washing with water will convert the precipitate into the pure oxymercuric hydrogen salt. After digesting with the mercury nitrate, the first washing waters used must contain a little nitric acid to guard against the formation of any oxynitrate.

Oxymercuric sodium imidosulphonate when long washed with much water leaves a much smaller weight of oxymercuric hydrogen

salt, and imparts continuously to the wash-waters small quantities of a mercury sodium imidosulphonate—apparently the same as that got by digesting mercuric oxide in a solution of mercuric disodium imidosulphonate and perhaps $O[HgN(SO_3Na)_2]_2$. These waters are neutral or faintly alkaline, and when evaporated a little on the water-bath yield small quantities of micaceous crystals. This reaction appears to be expressed by the equation :—

$$2Hg(OHgSO_3)_2NHgN(SO_3Na)_2 + OH_2 = O[HgN(SO_3Na)_2]_2$$
$$+ 2HN(SO_3HgO)_2Hg.$$

Oxymercuric hydrogen imidosulphonate is an anhydrous salt. Already dried in an ordinary desiccator, it loses, when further dried at 100° or above, 0·7 to 0·8 per cent. of water. It may be heated in dried air to 180° or higher without changing. Only a little below a dull red heat does it decompose, and then slowly gives water, nitrogen, and sulphur dioxide. At this temperature it is yellow, but when cooled it resumes the white colour of the undecomposed salt. At the softening point of hard glass it melts to a dark red liquid and effervesces, yielding sublimates of mercury metal, mercurous and mercuric sulphates, and some other mercuric salt not sulphate and apparently nitrogenous. It seems impossible, even in a vacuum, to decompose it completely before the mercury sulphates themselves partly decompose. The first gases given off consist of nitrogen with half its volume or more of sulphur dioxide—those coming after still contain nitrogen along with sulphur dioxide and oxygen. When rapidly raised to a red heat, the salt, at the moment of melting, effervesces almost explosively.

Its reactions with sodium hydroxide and chloride, and with trisodium imidosulphonate are similar to those of oxymercuric sodium salt (p. 106). Its basic or oxysalt character is thus clearly demonstrated, independently of the evidence from its quantitative analysis.

The presence of hydrogen, not merely as water of hydration, is shown by first heating the salt for hours at 180° in a current of dried air, and then afterwards getting, at a much higher temperature, water vapour condensed (containing sulphuric acid). As the salt is calculated to yield only a little more than one-thousandth of its weight of hydrogen, and the isolation of the water from all mercury would be difficult, hydrogen was not quantitatively estimated.

The analytical results obtained were :—

	$HN(SO_3)_2Hg_3O_2$	(a)	(b)
Mercury	74·35	74.03	74·09
Sulphur	7·93	8·06	8·00
Nitrogen	1·73	1·76	1·65
Hydrogen	0·12	—	—
(Sodium)		0·06	0·06
(Moisture)		0·20	0·13

The preparation, *(a)*, was made by the principal and direct method. The other, *(b)*, was digested with mercuric nitrate to remove sodium which it at first contained, and then washed with water slightly acidulated with nitric acid.

On writing down the reaction by which the salt under consideration is formed from mercuric nitrate and trisodium imidosulphonate:—

$$3Hg(NO_3)_2 + Na_3N(SO_3)_2 + 2OH_2 = HN(SO_3)_2Hg_3O_2 + 3NaNO_3 + 3HNO_3$$

—and comparing it with the equation where oxymercuric sodium imidosulphonate is the product, it will be seen that in both cases the nitrate comes out half as sodium nitrate and half as nitric acid, and that the oxymercuric hydrogen salt may be represented as resulting from the reaction of the sodium salt with mercuric nitrate, thus :—

$$Hg(NO_3)_2 + NaN(SO_3)_2Hg_2O + OH_2 = HN(SO_3)_2Hg_3O_2 + NaNO_3 + HNO_3.$$

Now this reaction can actually be realised, as already stated, but only with difficulty and in the presence of great excess of mercuric nitrate solution, active probably by virtue of its free acid. In the direct method of preparing the hydrogen salt, it is formed simultaneously with half as much again of nitric acid as is produced when the sodium salt is formed. From this it would appear that there is a point—difficult to determine by direct experimentation with nitric acid, because of its liability to cause hydrolysis after a time, and of the uncertainty in knowing when the sought-for change has occurred—where, the nitric acid being in sufficient quantity along with the mercuric nitrate, nitrogen takes or keeps hydrogen in place of mercury, and the second half of the sulphonic radicals, as well as the first, takes mercuric oxide in place of sodium, just as it does in other cases even in presence of free acid.

Mercury sodium imidosulphonates.—There are two mercury sodium imidosulphonates. The monosodium salt is oxymercuric or basic, but the disodium salt is normal, and corresponds in composition with the potassium salt obtained by Berglund.

Unless gradually added, mercuric nitrate causes, almost immediately, a white crystallo-flocculent precipitate in solutions of the trisodium imidosulphonate, which disappears on agitation so long as enough of the sodium salt remains to keep the mixture alkaline or neutral. When this point is passed, precipitation is permanent, and *oxymercuric sodium* salt is the product. When the nitrate is only added till neutrality is reached or nearly reached, the liquor soon begins to form small brilliant crystals, or else will do so after some evaporation. These crystals are *normal mercuric disodium imidosulphonate.* Adding solution of trisodium imidosulphonate to that of mercuric nitrate, free from any unnecessary excess of nitric acid, causes again, as already stated, a white crystallo-flocculent precipitate which, with sufficient

excess of nitrate remaining, is *oxymercuric hydrogen salt*. With too little nitrate remaining, some oxymercuric sodium salt will be deposited. Mercurous nitrate behaves in the main like mercuric nitrate but gives besides a precipitate of the metal.

Mercuric disodium imidosulphonate.—Little more needs to be stated concerning the preparation of this salt from the trisodium salt. Here, as in other cases, the mercuric nitrate solution is preferably to be highly concentrated, because then the excess of nitric acid necessary becomes very small. To form such a solution, moderately concentrated nitric acid should be so far saturated with mercuric oxide that oxynitrate begins to form, and then decanted from excess of oxide and left to clarify by subsidence. When too much mercuric nitrate has been added to the sodium salt, the proportions can be rectified quite successfully by adding more sodium salt. The mercuric disodium salt can be purified if necessary, by recrystallisation from hot water.

Mercuric disodium imidosulphonate can also be readily prepared from disodium imidosulphonate and mercuric oxide. The two substances may be triturated together in about the right proportions mixed with water, and then warmed with it. The solution is to be filtered if necessary, and then set aside to crystallise.

The crystals of mercuric disodium imidosulphonate are small brilliant prisms, always separate, quite permanent in the air, and sparingly soluble in cold water. The solution has a neutral reaction. The crystals contain six molecules of water of which only four are lost in a vacuum at common temperatures.

Heated to 100°, after exposure in a vacuum desiccator, it loses most of the remaining water, but not all, for then (and at higher temperatures, such as that of 130°, more quickly) it also increases slowly in weight by fixing atmospheric moisture, becoming hydrolysed and thereby strongly acid (see effects of heating dipotassium imidosulpho-

nate, p. 64). Heated more strongly in an open tube, it yields a small sublimate of an ammonia-sulphite salt, mercurous sulphate, and mercury, along with sulphur dioxide and nitrogen as gases, and mercurous, mercuric, and sodium sulphates as residue.

Heated slowly in the vacuum of the Sprengel-pump, it suffers change in a way that can be more closely studied. Even at 444° no material alteration takes place in the salt, but just below a red heat, it decomposes steadily, temporarily blackening through formation of mercuric sulphide, and giving much mercury as a sublimate, nitrogen and sulphur dioxide in the proportions of two volumes of the former to three of the latter, a little, very volatile, white ammoniacal sublimate along with a very little water, and another white sublimate volatilising again at 350-400.° The residue is sodium sulphate. The margin, remote from the heat, of the sublimate last mentioned consists of mercurous sulphate, but the rest of this sublimate, which adheres to the glass firmly, is of a peculiar nature but imperfectly made out. It is, however, a mercury compound scarcely affected by potassium hydroxide, boiling water, or dilute nitric acid, and is apparently a nitrogen-holding derivative of mercuric sulphate.

The main changes by heat appear to be—

$$2HgN_2(SO_3Na)_4 = 4SO_4Na_2 + HgSO_4 + HgS + 2SO_2 + 2N_2,$$

and then by further heat—

$$4SO_4Na_2 + 2Hg + 4SO_2 + 2N_2.$$

The ammoniacal sublimate and moisture are evidently due to water retained by the salt, probably through hydrolysis; while mercurous sulphate will have come from mercury and mercuric sulphate.

With solutions of ordinary metallic salts, the mercuric disodium salt gives the various double mercury imidosulphonates noticed by Berglund. For example, with barium chloride it gives a precipitate of mercury barium imidosulphonate,

Sodium hydroxide precipitates mercuric oxide from the pure salt, but not in the presence of sodium imidosulphonate, and under any circumstances the precipitation of the mercury is far from complete. According to Raschig, Berglund found that mercuric dipotassium imidosulphonate gives no precipitate with potassium hydroxide, but we find that, in this respect, the potassium salt behaves like the sodium salt, except that the precipitation is perhaps less. Dilution lessens precipitation (see behaviour of mercuric oxide with trisodium imidosulphonate, p. 73).

Ammonia gives a white precipitate. So, too, does ammonium chloride, which in this case is probably amidomercuric chloride. Mercuric oxide dissolves slightly in solution of the mercury sodium salt, rendering it alkaline. Nitric acid and, still more so, hydrochloric acid dissolve the salt freely. Nitric acid does not immediately decompose it apparently, but the action of hydrochloric acid effects the complete decomposition of the salt. If the quantity of this acid is insufficient for the whole of the salt, its action is confined to its equivalent quantity, the rest of the salt being left undissolved, and no preferential replacement of the sodium or the mercury by hydrogen taking place. By extraction with ether, by evaporation, and by other tests, the change effected has been ascertained to be the formation of mercuric sodium chloride, sodium acid sulphate, and amidosulphonic acid.

Analysis has given the following results :—

	$HgN_2(SO_3Na)_4,(OH_2)_6$	Found.
Mercury	26·74	27·16
Sodium	12·30	12·35
Sulphur	17·11	17.19
Nitrogen	3·74	3·95
Water	14·44	13·15 {8·86 lost *in vac.* / 4·29 „ at 100°

The exposure in the vacuum was for 40-45 hours. The further loss at 100° was the greatest obtainable in undried air, longer heating being followed very slowly by increase of weight. As analysed, the salt shows nearly a tenth less than the water calculated. Efflorescence of the sample was not noticed, but may have occurred to a slight extent. The main cause of the deficient finding is without doubt, however, fixation of some of the crystallisation-water or of atmospheric moisture by hydrolysis.

Oxymercuric sodium imidosulphonate.—In the preparation of this salt, when the sodium imidosulphonate is taken in double molecular proportion to the mercuric nitrate, that being the calculated proportion, the process works well. The highly acid mother-liquor retains much of the salt in solution, but will let it fall if nearly neutralised with sodium hydroxide, and then contains scarcely any other mercuric or imidosulphonic salt. There is, however, no necessity to adhere closely to the calculated proportions. Provided that the sodium salt is neither in quantity great enough to redissolve the precipitate as mercuric disodium salt nor so small as to leave the nitrate in much excess, the process will succeed; but it is better to use too little than too much of the nitrate, above all should its solution contain any quantity of free acid. Any precipitation which water alone would cause of oxynitrate in the solution of mercuric nitrate, is prevented by the presence of the sodium imidosulphonate, since this salt forms its basic mercury derivative by liberating half the nitric acid of the nitrate that it decomposes, and this acid is much more than enough to keep any remaining nitrate from passing into oxynitrate, and, consequently, the washed precipitate proves to be always free from nitrate. It may be washed sufficiently with water, which only very slightly acts upon it, and may then be dried, either on paper or better on a tile.

On attempting to form a third and intermediate mercury sodium

salt, which was to have the formula, $O[HgN(SO_3Na)_2]_2$, analogous to the oxynitrate, $O(HgNO_3)_2$, we got, instead, only the other two salts, one in solution, the other as a precipitate. The oxymercuric sodium salt was thus obtained from a liquor which from first to last was never acid, even the mother-liquor being still faintly alkaline. The sulphonate and nitrate were used in the proportions—$2NaN(SO_3Na)_2$ to $Hg(NO_3)_2$; the nitrate was added gradually, with stirring, to the dilute solution of the sulphonate, and simultaneously a solution of sodium hydroxide run in in the proportion of nearly five-sixths of a molecular quantity. By this procedure, much yellow mercuric oxide was formed along with the white precipitate, but by prolonged stirring, the precipitate lost all tinge of yellow. The mother-liquor now contained much disodium mercury imidosulphonate, and the precipitate, washed and drained on a tile, proved to be pure oxymercuric sodium imidosulphonate (analysis, *d*). Allowing for sodium hydroxide used in neutralising the free nitric acid present in the mercuric-nitrate solution, the reaction appears to be that expressed by the equation :—

$$6NaN(SO_3Na)_2 + 6Hg(NO_3)_2 + 4NaOH =$$

$$= 2HgN_2(SO_3Na)_4 + Hg(OHgSO_3)_2NHgN(SO_3Na)_2 + 12NaNO_3 + 2H_2O.$$

Oxymercuric sodium imidosulphonate contains water of crystallisation and appears to be a little efflorescent ; otherwise, it is for a long time permanent, wet or dry, though ultimately becoming yellow-brown by decomposition. Protracted washing with water decomposes it, as does also digestion with concentrated mercuric-nitrate solutions, a residue being left in both cases, which is the hydrogen oxymercuric salt (p. 97). It is very stable when but moderately heated, only losing its water, with the exception of a very little. At about 135° it permanently changes colour slightly, and at a much higher temperature melts to a dark red-brown liquid and effervesces, decomposing in

a way essentially the same as that followed by the normal mercury sodium salt. Most gradual heating in a current of dried air to a temperature of 170° fails to expel more than four-fifths of the water, in consequence no doubt of hydrolysis.

It is much more readily dissolved by hydrochloric acid than by nitric or sulphuric acid. From its hydrochloric-acid solution it can not be recovered by neutralisation, being almost instantly decomposed, like the normal mercuric sodium salt. It is converted by sodium hydroxide into mercuric oxide insoluble, and mercuric disodium imidosulphonate dissolved. Its basic composition is at once demonstrated by the action on it of sodium chloride, which leaves insoluble mercuric oxide, and dissolves the rest, probably, as the two salts, mercuric sodium imidosulphonate and mercuric sodium chloride. Trisodium imidosulphonate dissolves it, but not to a large extent. When the solution of this salt is concentrated, a little mercuric oxide may separate, but enough water added causes this gradually to dissolve. When the trisodium salt is added to the oxymercuric sodium salt still in its mother-liquor, free dissolution at once occurs; but without the mother-liquor the reaction is as above stated.

The composition of the salt is expressed, as the following analytical results show, by the formula—$OHg_2N(SO_3)_2Na,(OH_2)_2$—which has, however, to be doubled to display its constitution. The water comes out low, partly because of loss by efflorescence, but mainly by getting fixed through hydrolysis.

	Calc.	(*a*)	(*a*)	(*b*)	(*c*)	(*d*)
Mercury	61·63	62·18	61·22	61·56	—	61·73
Sodium	3·54	3·87	4·14	3·97	3·57	3·56
Sulphur	9·86	10·23	10·06	10·03	—	9·79
Nitrogen	2·16	—	2·27	2·24	—	—
Water.	5·55	—	4·44	—	4·88	4·58

The preparation, (*a*), was precipitated from three molecules of mercuric nitrate by two of imidosulphonate, while (*b*) was got by adding sodium hydroxide to the decanted mother-liquor of (*a*). The preparation, (*c*), was precipitated from two molecules of nitrate by one molecule of imidosulphonate, and (*d*) was formed in a non-acid mother-liquor. All preparations, as well as their mother-liquors, were free from sulphate.

The formation of oxymercuric sodium imidosulphonate is expressed by the equation :—

$$4Hg(NO_3)_2 + 2NaN(SO_3Na)_2 + 2OH_2 = Hg(OHgSO_3)_2NHgN(SO_3Na)_2 + 4NaNO_3 + 4HNO_3.$$

This reaction is noticeable for being one in which sodium is withheld in the precipitate from nitric acid ; but, if imidosulphonic acid be regarded as a weaker acid than nitric acid, the precipitation of potassium nitrate by tartaric acid or perchloric acid is quite as remarkable, while if, as is most probable, it is like sulphuric acid, then the retention of sodium from the nitric acid is only natural. It is hardly noticeable for the precipitation of a basic salt with production of nitric acid, since similar reactions are common with oxygenous mercuric salts.

In its constitution, as regards the oxylic mercury, oxymercuric sodium imidosulphonate resembles oxymercuric sulphate and oxymercuric sulphite, as the following formulæ display (*J. Coll. Sci.*, **1**, 101) :—

Imidosulphonate	$(NaO{\cdot}SO_2)_2{:}N{\cdot}Hg{\cdot}N{:}(SO_2{\cdot}O{\cdot}Hg{\cdot}O)_2{:}Hg$
Sulphate	$Hg{:}(O{\cdot}Hg{\cdot}O{\cdot}SO_2{\cdot}O{\cdot}Hg{\cdot}O)_2{:}Hg$
Sulphite	$Hg{:}(SO_2{\cdot}O{\cdot}Hg{\cdot}O)_2{:}Hg$

Constitution of the Mercury Imidosulphonates.

From the widely established character of the relation between

mercury and the nitrogen of ammonia and cyanogen, and from that of the mercury in its oxygenous salts, particularly those of sulphuryl, such as sulphites and sulphates, the relations of mercury in its imidosulphonates become of much interest, as likely to vary from those of other basylous elements.

Firstly, as to the mercury disodium and mercury dipotassium imidosulphonates, no hesitation will be felt in accepting for these salts—say that of sodium—the formula, $HgN_2(SO_3Na)_4$ (Berglund), yet this point is not so simple as it seems. Calcium forms the salt, $CaNaN(SO_3)_2$; silver forms the salt, $Ag_2NaN(SO_3)_2$; and mercury itself the salts, $OHg_2NaN(SO_3)_2$ and $O_2Hg_3HN(SO_3)_2$. All these salts have only one-third of the bases either sodium or hydrogen, and it will be well, therefore, to briefly review the reasons for writing $HgN_2(SO_3Na)_4$.

Disodium hydrogen imidosulphonate is a salt neutral or only slightly acid to litmus, although active as an acid, and therefore, it cannot hold the group, SO_3H, since this always gives to its compounds sourness and strong action on blue litmus. It must therefore be written $HN(SO_3Na)_2$. Mercury readily takes the place of the hydrogen of this salt, or of one atom of the sodium of the alkaline trisodium imidosulphonate, and the resulting mercury disodium salt is neutral in reaction. Transposition of the metals cannot be admitted to take place in its formation for two reasons. One is that the alkaline reaction of the trisodium salt disappears when it becomes the mercury sodium salt, and there is no accounting for this if the mercury displaces the sulphonic sodium. The other reason is that if the mercury takes the sulphonic relation in the salt, there is to be seen in this salt an exception to the observation that all oxylic mercuric salts existent in presence of water are insoluble basic or oxysalts.

This theory of the constitution of the normal mercury double sulphonates accords with the interesting behaviour of these salts

towards acids and alkalis. The latter only partially precipitate mercuric oxide from them, because the mercury is in immediate relation with (unoxidised) nitrogen. Nitric acid, which can replace the mercury by hydrogen, cannot do the same with the potassium or sodium, because this is in oxylic relation with sulphuryl, as it is in sulphates, and therefore irremovable by this acid. Sulphuric acid also first replaces the mercury by hydrogen, when acting on the mercury potassium salt, although from the mercury barium salt it takes first the barium away.

Berglund would have chemists regard mercury imidosulphonates as salts of an independent acid, in which mercury is combined with special force. A fuller knowledge of the imidosulphonates does not tend to support him in this. There are other series of double imidosulphonates besides that of mercury, apparently the only one observed by him; normal mercury hydrogen imidosulphonate has less stability than imidosulphonic acid itself; oxymercuric hydrogen imidosulphonate is a far more stable body than it; dipotassium and disodium hydrogen imidosulphonates have equal or greater claims to be treated as particular acids; lastly, it is highly probable that, powerfully as mercury takes the place of hydrogen in ammonia itself, it will have little of that power when two-thirds of that hydrogen have already been replaced by sulphonic radicals.

The constitution of the salt in which two-thirds of the base of the trisodium salt are replaced by mercury next requires attention. It might be treated as having the single atom of sodium in the odd basylous position, that is, united to the nitrogen; but against this four objections present themselves. One is that it is quite against probability that the sodium should hold the imidic relation rather than the sulphonic. A second is that it is quite as improbable that a mercury atom should be united half to nitrogen and half to oxygen,

while it is usual to find it thus united at once to oxygen and to an oxygenous radical. A third objection is that, whereas in other cases sulphuryl takes one-and-a-half atoms of mercuric oxide to saturate either of its valencies, it does not do so in this salt unless the sodium is given to one of the sulphuryls of the salt. The fourth objection lies in the great improbability that sodium in the imidic relation would resist, as it here does, displacement by hydrogen on contact of the salt with nitric acid. Avoiding these four difficulties by placing one-fourth of the mercury with the nitrogen, there remain the usual one-and-a-half atoms to unite with half one of the sulphuryls. The relations of the salt, and its constitution as here developed, require that its formula should be double that expresed in the lowest terms, in order to allow of the oxylic mercury being shown apart from the non-oxylic—$Hg:(OHgO{\cdot}SO_2)_2:N{\cdot}Hg{\cdot}N:(SO_3Na)_2$.

It becomes now clear that the sodium resists the action of nitric acid because it is in oxylic relation to sulphuryl, as pointed out in discussing the constitution of the normal mercury disodium salt ; that nitric acid removes a fourth of the mercury, replacing it by hydrogen, because this fourth part is in relation to the nitrogen ; that nitric acid does not remove the rest of the mercury because this exists as the oxymercuric group found in oxylic relation with sulphuryl in sulphate and sulphite, also then resisting the action of nitric acid ; and, lastly, it becomes clear how it is that the oxymercuric hydrogen imidosulphonate has quite consistently a constitution different from that of the oxymercuric sodium salt, and how the one salt is formed from the other. (Cf. Divers and Shimidzu on *Mercury Sulphites*, *J. Coll. Sci.* **1**, 101).

In fact, the constitution of the oxymercuric hydrogen imidosulphonate follows obviously from the production of the salt by (acid) mercuric nitrate in excess. From what we actually can observe in the

case of the normal mercury dipotassium salt we know that nitric acid of itself should act as represented by this equation—

$$Hg(OHgOSO_2)_2NHgN(SO_3Na)_2+2HNO_3=$$
$$=Hg(OHgOSO_2)_2NH+HN(SO_3Na)_2+Hg(NO_3)_2$$

—replacing the imidic mercury by hydrogen, but not touching the oxymercuric group, in conformity with its inability in other cases (sulphites and sulphates) to do so when the group is joined to sulphuryl. In the absence of mercuric nitrate this reaction is slowly followed by hydrolysis of the disodium hydrogen salt through the unavoidable excess of nitric acid, but in presence of mercuric nitrate hydrolysis does not take place. The mercuric nitrate finishes the formation, just formulated, of the oxymercuric hydrogen salt, in the way shown by the equation—

$$HN(SO_3Na)_2+3Hg(NO_3)_2+2OH_2=HN(SO_2OHgO)_2Hg+2NaNO_3+4HNO_3.$$

—from which equation it will also be sufficiently clear how the whole change can be effected by mercuric nitrate without any addition of nitric acid.

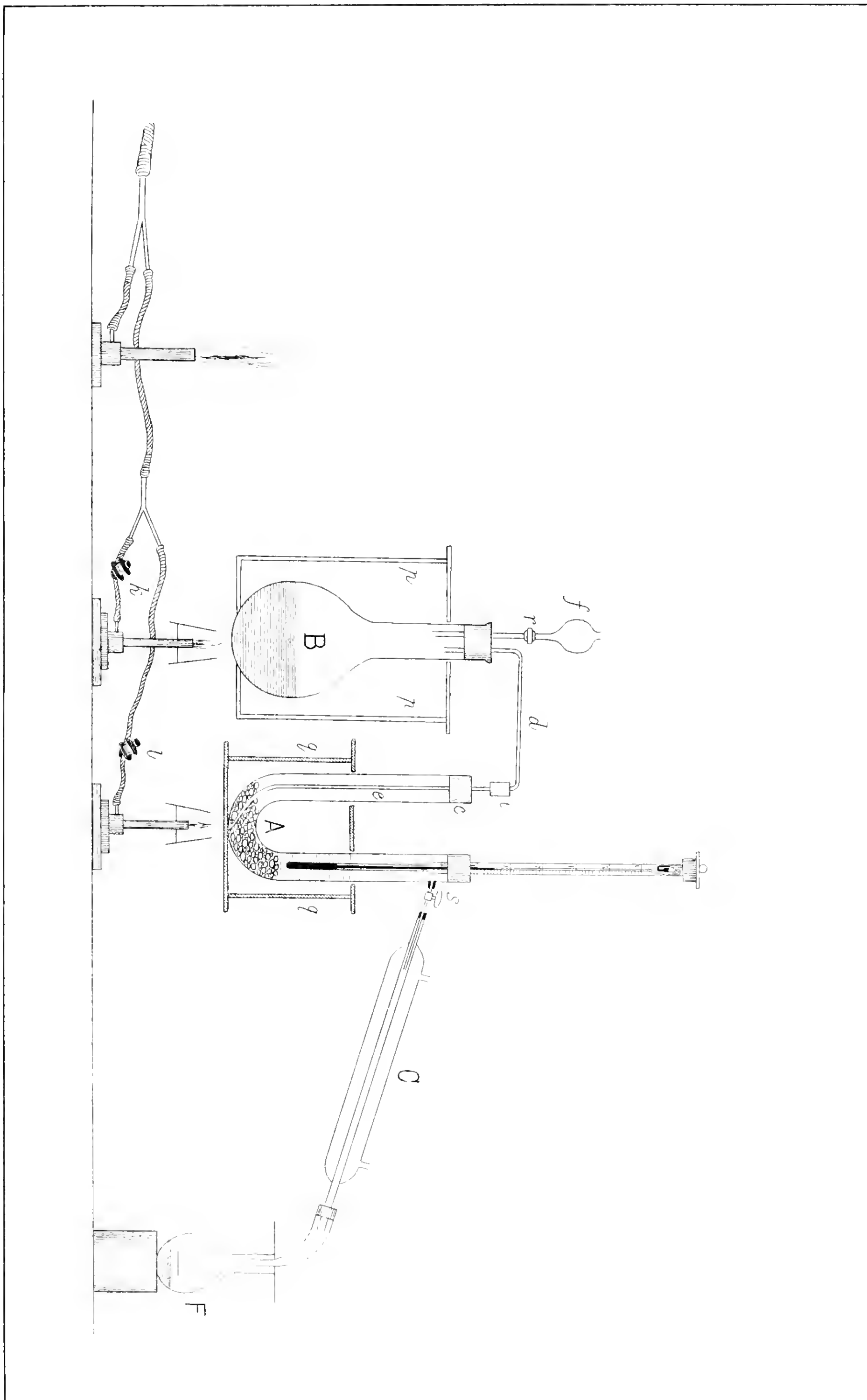
From the constitution given to this salt may be seen why it can, so remarkably, be left for days in a nitric-acid solution of mercuric nitrate without hydrolysing. For hydrolysis can only occur when some of the sulphonic group becomes acid, or SO_3H, and here the nitric acid, especially in presence of much mercuric nitrate, is powerless to displace the oxymercuric group by hydrogen.

There remains now only to tabulate the three mercuric salts, the constitution of which has been discussed, in order to bring out their relations quite clearly, and particularly the intermediate relation of the oxymercuric sodium salt.

Imidosulphonate.	Formula.
Mercuric disodium	$Hg:\begin{matrix}N(SO_3Na)_2\\N(SO_3Na)_2\end{matrix}$
Oxymercuric sodium	$Hg:\begin{matrix}N(SO_3Na)_2\\N(SO_3HgO)_2Hg\end{matrix}$
Oxymercuric hydrogen.	$HN(SO_3HgO)_2Hg$

Addendum.

Oxyamidosulphonates.—In our last paper, we had to call attention to the discordance between some of the results of the work done by others and by ourselves on *sulphazotised salts.* We are glad to be able to bring in support of the accuracy of certain of our own statements the testimony of one of the other workers, Hr. Dr. Raschig, who wrote soon after the appearance of the paper on oxyamidosulphonates to us request when next publishing to make known that he now entirely agrees with our account of the decomposition of oxyamidosulphonates bycaustic alkali (*J. Coll. Sci.*, **3**, 218), and withdraws his own statement concerning it, which was based upon qualitative reactions only.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On the Anatomy of Magnoliaceæ.

By

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With Plates II-V.

Introductory Remarks.

The investigation of the present subject was begun in the autumn of 1890, at the suggestion of Prof. J. Matsumura, and was continued more than a year. During that time I received much useful advice from him, and also from Prof. R. Yatabe, to both of whom I am much indebted. The object of my researches was to find out what anatomical peculiarities characterize the Magnoliaceæ as a whole; what distinctive characters are presented by each of the different groups included in it; and to what extent all the species of it may be anatomically distinguished from one another.

These questions naturally present themselves, if we remember that systematic botanists in recent times have not agreed as to the limits of this family, and have treated its members in different ways. Some botanists extend its province by including certain tribes not usually admitted, while others restrict it by omitting tribes commonly included. Such indefiniteness as to its extent shows that this natural family is not very natural, and I hope that the study I have made of its anatomical characters may be of some service in remedying this defect.

I examined all the species of Magnoliaceæ that were accessible, including those placed in other families by some botanists. Such being the case I find it more convenient to use in my dissertation the term Magnoliaceæ in its wider sense. It also best answers my purpose to divide this family, as is done by Luerssen[1] and others, into four tribes; namely :—

Magnolieæ,	*Schizandreæ*,
Illicieæ,	*Trochodendreæ*.

The number of genera included in these tribes does not exceed fourteen, on the highest estimation, and that of the species known at present ranges between seventy and eighty. However, I could examine only twenty-four species and two varieties. Although the number of the species examined is small when compared with that of all the species known, yet those examined are distributed among ten genera, which are in their turn distributed among the four tribes. Therefore, I am perhaps right in believing that the anatomical characters of the species I examined represent fairly those of the whole family.

With the exception of two dried specimens in the herbarium of the Science College, which I was allowed to examine, the materials for my study were mostly obtained in the University Botanic Garden in Koishikawa. As I could get no specimens of the main root proceeding immediately from the seedling, but only of younger secondary roots, I examined these branchlets in order to get some knowledge of the anatomical characters of the young root, such as the arrangement of the xylem-plates, &c.

It will be well, I think, to define at once a few words which though often convenient to use in describing the structure of the plant-body, are yet somewhat vague in their meaning. "Sclerenchy-

1. Luerssen, *Grundzüge der Botanik*.

matous fibre" is used in the same sense as bast-fibre, and "sclero-blast" in that of stone-cell; when either indefinitely is to be denoted, the expression "sclerenchymatous element" is often used. By "sclerenchymatous sheath" I mean the single mass of sclerenchymatous fibres, or the numerous isolated groups of them, which lie at the external limit of the fibro-vascular bundles and form a ring either continuous or interrupted at intervals. I consider this sheath to be a part of the bundles, and not an independent structure lying outside them. By "cortex" I mean all the tissues which lie outside the cambial zone. That part of it which corresponds to the phloëm (including the sclerenchymatous sheath, when this exists), I generally call the "inner cortex"; and all the parts outside this, the "outer" or "external cortex." In the case of very young roots I mean by "cortex" all the tissue lying outside the endodermis.

In the following pages the anatomical characters of each genus will be first described, and then compared. It might appear sufficient to have compared at once the characters of the different genera without giving a special description of each genus, but this would have led to confusion, as well as to the omission of many interesting isolated facts.

Anatomical Characters.

Under the headings, *Stem*, *Petiole*, *Blade*, and *Root*, the points of structure peculiar to each are described, while under the first are given also those points common to all four.

Tribe I. Trochodendreæ.

This tribe consists of three genera, species of each of which I examined.

Euptelæa.

E. polyandra, Sieb. et Zucc., is the only species of this genus which I examined. It is a small tree, found in many parts of Japan, in which the aromatic property so common in Magnoliaceæ is entirely absent.

Stem.—The epidermal cells present no peculiarity ; the cuticle is not well developed. There is found on the epidermis a number of lenticels, which appear to the naked eye as white specks. Cork is developed immediately beneath the epidermis, its cells being of comparatively large size, while those forming the other tissues are generally very small. The hypoderma is represented by a layer of somewhat thick-walled parenchymatous cells beneath the cork. Sclerenchymatous elements of any kind are totally absent in the outer cortex ; but there are found in it many sacs, each of which contains an aggregate of crystals of calcium oxalate (Pl. II, Fig. 2).

The sclerenchymatous ring which accompanies the fibro-vascular bundles is well developed, but is interrupted at the points where the large medullary rays run radially through the phloëm (Pl. II. Fig. 1). Sclerenchymatous elements are absent in the inner phloëm ; but a few scleroblasts find their way into those portions of the phloëm rays which run near the sclerenchymatous ring (Pl. II. Fig. 1, *s*). In specimens collected while the cambial zone is in activity, there is seldom seen any marked distinction between that zone and the xylem, gradual transition taking place from lignified cells to unlignified ones. (Pl. II. Fig. 1). In the phloëm portion sieve-tubes are distinctly to be seen, especially in longitudinal sections. The xylem contains vessels, tracheïds, fibres, and wood-parenchyma. The vessels generally have fibrous markings on their walls. In those portions of the xylem which border on the pith and constitute the medullary sheath there are found, in young specimens, groups of elongated cells with

thin unlignified walls. Each group surrounds the first-formed spiral vessels and is destined to be lignified in the course of time. Primary medullary rays usually consist of two or three radial rows of cells; besides these there are found numerous small rays consisting of a single row of cells.

The pith commonly consists of cells with thick, lignified, pitted walls, but sometimes there remains in the central portion of the pith a group of cells with unlignified walls.

Petiole.—On the epidermis a few hairs are sometimes present. They especially grow in the groove which runs longitudinally along the upper surface of the petiole. The hypoderma is present. The fibro-vascular bundles form a ring enclosing the pith. Sclerenchymatous fibres are well developed and constitute a continuous sheath around the bundles; they are unlignified in the basal portion of the petiole. The phloem portion is here and there crossed by the medullary rays, and the portion of the ray which bridges the phloëm is made up of scleroblasts. Unlike that of the stem the pith is formed of thin-walled parenchyma.

Blade.—Its structure is generally compact. The stomata do not present any peculiarity. Hairs grow on the midrib and their base is often made up of several cells. The cuticle of the midrib forms a number of longitudinal ridges, which present in a cross section cuticular indentations (Pl. II, Fig. 3). Directly under the epidermis of the lower side of the blade there is found a layer of cells which forms the hypoderma. The arrangement of the fibro-vascular bundles of the midrib is not completely circular, but shows a discontinuous portion turned towards the upper surface of the blade. Aggregates of crystals are also met with in the cells of the cortical portion of the midrib.

Root.—In the older root the general structure does not differ

much from that of the stem, except that pith is absent. The sclerenchymatous ring contains both short and elongated elements. In the young root hairs are copiously found on the epidermis. The arrangement of xylem-plates seems to belong to the diarch type (Pl. II. Fig. 4, A). As the root becomes a little older the two xylem-plates which are opposite in their position become united, thus forming an elliptical mass of wood (Pl. II. Fig. 4. B). The endodermis remains cellulose as long as it exists.

Cercidiphyllum.

Of this genus I examined a single species, *C. japonicum*, Sieb. et Zucc. The plant is found in the mountainous parts of this country. It attains a great height and yields a valuable timber. It is not aromatic though the opposite statement is made by some authors.[1]

Stem.—The tissue elements of this plant are very small in size; the compactness of texture of the wood is owing to this fact. The epidermis is early shed, so that except in very young shoots it is completely absent. The cork is very compact like the other tissues. The hypodermal layer is present under the cork. Cells containing prismatic crystals of calcium oxalate are abundantly found both in the outer cortex and in the phloëm, and are especially numerous in the phloëm rays (Pl. II. Fig. 5). The occurrence of these crystals is so common that I found them in various specimens collected in different seasons.

The groups of sclerenchymatous fibres are well developed and form bands alternating with the soft bast; but there does not exist a continuous sheath of sclerenchyma enclosing the fibro-vascular bundles. Sieve-plates are seen in the phloëm, though not very distinctly. The

1. Baillon, *Natural History of Plants*, (translated from the French).
Engler and Prantl, *Die natürlichen Pflanzenfamilien*.

xylem contains tracheæ, fibres, and wood-parenchyma. Bordered pits are found in the walls of the vessels; they appear in a surface view like a slit encircled by a halo, and not like an encircled spot as commonly seen in the wood of *Pinus*. Most of the medullary rays consist of single rows of cells.

The parenchymatous cells of the pith have thick lignified walls, even in the young specimen, and contain an ample quantity of starch.

Petiole.—The epidermis consists of a single layer of cells, and the cuticle is not well developed. The hypoderma is represented by a layer of thick-walled parenchymatous cells. There is present in the external cortex no sclerenchymatous element, but deeper in this region there lies a zone of tissue presenting an irregular appearance, due to the distortion of parenchymatous cells (Pl. II, Fig. 6, *a*). The fibro-vascular bundles are arranged in a circle, as in the stem, and enclose the pith in the centre (Pl. II, Fig. 6). The bundles are encircled by a well-developed sclerenchymatous sheath, which is interrupted by parenchymatous cells only at a few points (Pl. II, Fig. 6). The fibres are unlignified in the basal portion of the petiole. Prismatic crystals are found in the phloëm. The pith parenchyma unlike that of the stem has unlignified walls.

Blade.—Its general structure, as also the form of its stomata, has no peculiarity. There are several main-ribs instead of a single midrib, and their fibro-vascular bundles present a semi-circular arrangement, while those of the petiole are circularly arranged. A group of thick-walled cells is found within that portion of the epidermis, which lies on the upper side of the main-ribs. Aggregates of crystals are found in the cortical region of the main rib. The cuticular ridges are not so conspicuous as in the midrib of *Euptelea*.

Root.—The structure of the older root is similar to that of the stem. Crystals are copiously found here also in the phloëm.

Sclerenchymatous fibres are also well developed. In the young root the endodermis is distinctly seen, and the cells that constitute it remain cellulose. The xylem-plates present the tetrarch arrangement, but sometimes also the triarch.

Trochodendron.

To the present genus belongs the single species, *T. aralioides*, Sieb. et Zucc. It is a tree destitute of any aromatic property. It is grown in many parts of Japan. Articles of various shapes are made from its wood by means of the turning-lathe. Its bark yields bird-lime.

Stem.—The epidermis is of the usual structure. The cuticle is very well developed, and striæ, which are perpendicular to the cuticular surface, and coincident with the boundary lines between the epidermal cells, are distinctly seen. Cork is present immediately beneath the epidermis. The hypoderma is represented by a layer of somewhat thick-walled, closely arranged parenchymatous cells. The portion of the outer cortex which lies within the hypoderma consists of a loose tissue with many interstices, and trichoblasts are here found in great abundance. They are of peculiar shape and are characteristic of the present species (Pl. III, Fig. 9); many-armed trichoblasts are found in a few other genera of Magnoliaceæ, such as *Magnolia* and *Michelia*, but these are much simpler in form.

The sclerenchymatous sheath is well developed and almost uninterruptedly encircles the fibro-vascular bundles. Besides these fibres the sheath contains short sclerenchymatous cells or scleroblasts. A great number of oil drops is found both in the external cortex and in the phloëm. With the exception of a few spiral tracheæ found in the primary wood and of the parenchyma forming the medullary rays, the whole xylem is made up of tracheïds. The walls of the latter

present bordered pits of various forms, more numerous on their radial than on their tangential surface; and round pits are mostly present on the tangential surface, while elliptical forms predominate on the radial surface. The primary medullary rays are very broad, consisting of several rows of cells.

The parenchymatous cells of the pith have thick lignified walls, which are pitted, and there is no trichoblast, though the opposite is stated to be the case by certain writers.[1]

Petiole.—The general structure is similar to that of the stem. A great many trichoblasts are found in the external cortex. The fibro-vascular bundles are so arranged as to present a somewhat semi-lunar form with its concave side turned to the upper side of the petiole (Pl. III, Fig. 7). The bundles are rather scattered in the basal portion of the petiole, but in the other portions they are more closely united. There exists at the external limit of the phloëm the sclerenchymatous sheath, but it consists of unlignified fibres.

Blade.—The contour of the epidermal cells is not wavy here as in the leaves of many other plants. The cuticle at the entrance of the stoma is elevated, and presents a cup-like appearance (Pl. III, Fig. 10). The palisade-parenchyma of the upper side of the blade consists of three or four layers of cells. Trichoblasts are met with in the region where the structure of the blade is loose, so that a number of intercellular spaces are left. The fibro-vascular bundles of the midrib are semi-circularly arranged, and a group of sclerenchymatous fibres is found in the cortical region lying on the bundles. The cuticular ridges of the midrib are wanting.

Root.—The general structure of the older root does not much differ from that of the stem. Trichoblasts are abundantly found in roots a little older, but their form is much simpler than that of those

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1 Engler und Prantl, *Die natürlichen Pflanzenfamilien.*

in the stem or leaf, and approaches that of the tricholblasts existing in the external cortex of some species of *Magnolia*. The hypoderma is absent. In the cortex of the young root there is found a number of cells with thick walls, slightly lignified (Pl. III, Fig. 8. *cs*). Some of these cells are short and others are elongated. They are not the young stage of the sclerenchymatous elements found in the older root, and are destined to be soon shed off with the parenchymatous cells among which they are scattered. The true trichoblasts come into existence at a much later time, while the sclerenchymatous fibres do not appear at all in the root, or only in somewhat old roots. The arrangement of the xylem-plates is of the diarch type (Pl. III, Fig. 8). The endodermis and pericambium are not well marked in the present species (Pl. III, Fig. 8).

Tribe II. Illicieæ.

Two genera, *Illicium* and *Drimys*, were examined in this tribe.

Illicium.

In this genus I examined, two species; namely, *I. religiosum*, Sieb. et Zucc., and *I. Tashiroi*, Max. The former is a small tree with aromatic odour, and widely distributed in Japan. *I. Tashiroi* was discovered a few years ago in Riukiu or Loochoo by a Japanese botanist, Y. Tashiro, whose name the plant bears, and I examined only a dried specimen of it.

Stem.—The epidermis has no peculiarity and has no appendages. The cuticle is very well developed, and striæ, perpendicular to the cuticular surface, and corresponding to the limits of the epidermal cells, are distinctly seen, as in *Trochodendron*. In stems a little older cork is developed immediately beneath the epidermis. In the outer cortex there exists no layer of cells that can represent the hypoderma,

all the cells of this region being of similar form and, in most cases, having pitted thick walls. There is not found among them any kind of idioblasts, which abound in several other genera of the same family.

The phloëm portion of the fibro-vascular bundles is almost destitute of hard bast, and only at the external limit of the bundles are there found a few scattered sclerenchymatous fibres, which represent the sclerenchymatous sheath (Pl. III, Fig. 11, *sl*). The soft bast certainly contains sieve-tubes, but I was not able to detect them clearly. Oil-globules are often met with in the phloëm region, and are a probable cause of the characteristic odour which is given out when a branch of the plant is bruised. In the external cortex of *I. Tashiroi* certain cells are met with which seem to be a receptacle for an oily substance (Pl. III, Fig. 12, A); but in *I. religiosum* the existence of such cells can not be clearly detected, though some cells of the external cortex may be suspected to be of that nature. In the xylem there are found tracheæ, fibres, and wood-parenchyma. The medullary rays are very narrow, generally consisting of one or two rows of cells. In *I. religiosum* the demarcation of the annual rings is generally not very clear, owing to the fact that the xylem-elements formed at the end of autumn of the preceding year are similar in size to those which are formed at the beginning of the next spring.

The pith is surrounded by a medullary sheath, in which spiral tracheæ are found. The parenchymatous cells of the pith have thick walls which are often pitted. They are lignified in old specimens of *I. religiosum*; and in *I. Tashiroi* this is the case even in young specimens. According to Prantl[1] sclerenchymatous cells are found in the pith of Illicieæ, but I do not find them in my specimens.

Petiole.—The fibro-vascular bundles of the petiole do not form a

1. Engler und Prantl, *Die natürlichen Pflanzenfamilien*.

ring enclosing the pith, as they do in the stem ; but constitute a somewhat crescent-shaped group with its concave side turned to the upper surface of the petiole. The xylem portion is incompletely lignified as is proved by the action of reagents. In the phloëm no sclerenchymatous fibres are present. The bundles are partly enclosed by a row of parenchymatous cells containing starch-grains, which perhaps represent a bundle sheath. In all other respects the structure of the petiole is similar to that of the stem.

Blade.—The cuticle is well developed. The fibro-vascular bundles of the midrib are semi-circularly arranged, and a few sclerenchymatous fibres accompany the bundles. The cells forming the palisade-parenchyma lying at the midrib have their height diminished and become almost round. Resin sacs are not found at least in *I. religiosum.*

Root.—An old root almost resembles the stem in general structure. In the young root the arrangement of the xylem-plates is of a diarch type. The cells that constitute the endodermis remain cellulose. Oil-drops are scattered about in the cortex of the young root. No sclerenchymatous fibre is seen in the phloëm, even in old specimens.

Drimys.

I examined only a dried specimen of one species of this genus ; namely, *D. dipetala*, Fr. M., which had come from New South Wales, and I can only give a few points about its anatomical characters, since I have only a very imperfect knowledge of them.

In general structure this species resembles *I. religiosum*, but there exist some decided structural differences between them. Thus in *D. dipetala* a great number of resin-sacs are present in the outer cortex (Pl. III, Fig. 12, B) and even in the phloëm ; a well developed sclerenchymatous ring accompanies the fibro-vascular bundles, and

the xylem consists exclusively of tracheïds. The last character is only found in the present genus and in *Trochodendron* among Magnoliaceæ. The parenchymatous cells of the pith are lignified even in a young specimen, as in *I. Tashiroi*, but I have not here met with any sclerenchymatous cells scattered about, (as they are stated to be by Prantl). The leaf is almost sessile, and the fibro-vascular bundles of the very short petiole present a semi-circular arrangement, as in *Illicium*. The bundles are similarly arranged in the midrib. A number of resin-sacs are found in the mesophyll. In the epidermal cells of the leaf, crystals are often met with, which are probably calcium oxalate.

Tribe III. Schizandreæ.

This tribe consists of the following two genera:

Kadsura and Schizandra.

The two genera are so closely related in anatomical characters as to make it convenient to describe them together. Of *Kadsura* I examined *K. japonica*, L., and of *Schizandra* I examined *S. nigra* Max., and *S. chinensis*, Bail. The three are small climbing shrubs, found in several parts of Japan. When a cut portion of these plants is put in water it yields a large quantity of mucilage, which in the case of *K. japonica* is sometimes used for dressing the hair. The two species of *Schizandra* give out a sweet odour when any portion of them is bruised, but *K. japonica* gives out little or no odour, when similarly treated. The berries of *S. chinensis* are said to be edible.

Stem.—The epidermis consists of a single layer of cells, and the cuticle is not well developed. The hypoderma is not represented. The cork is well developed directly under the epidermis and is of the usual type. In the cortical parenchyma there are scattered about a few cells which contain oil globules (Pl. IV, Fig. 14, *ro*); these are pro-

bably the source of the characteristic odour which the plants give out when bruised. These cells are conspicuous in *Schizandra* by their size, being much larger than the cells of the surrounding parenchymatous cells, but in *K. japonica* the oil-cells are scarcely larger than the surrounding cells, and are, besides, seldom met with. The cells of the cortical parenchyma are sometimes pitted in *K. japonica* (Pl. IV. Fig. 14).

Sclerenchymatous fibres are generally found at the external limit of the phloem, constituting the sclerenchymatous sheath, but they are not much crowded, either forming only a thin layer or else small groups (Pl. IV, Fig. 13, *sl*). Some of these fibres are septate, as may be seen in *K. japonica* (Pl. IV, Fig. 21). Scattered about in the phloëm (and also in the external cortex) there are found peculiar cells with thick lignified walls and a narrow lumen (Pl. IV, Fig. 20). Some of these cells are elongated like sclerenchymatous fibres, others are very short, simply round in form, or else provided with arms, which they push out freely among the surrounding cells. The outer layer of their thickened wall contains a great number of granules of calcium oxalate, which present an angular configuration, without the definite form of true crystals. These peculiar elements are structurally nothing else than sclerenchymatous fibres or scleroblasts, holding a number of granules imbedded close to one another in their walls, and for convenience' sake I will hereafter call them "crystal-bearing sclerenchymatous elements." I do not think that there exists any essential difference between them and the so-called spicular cells of *Welwitschia mirabilis*, which have been described by several authors, and are stated to be of the same nature as the crystal-bearing fibres found in *Araucaria*, &c. If I am right, the interesting fact is supplied of two groups of plants which are remote from each other in systematic position containing a very similar element in their

tissues. In the phloëm of *Kadsura* and *Schizandra* there are found many large intercellular spaces, which are apt to be mistaken for some kind of large ducts, but are really passages of lysigenetic origin. They are on all sides bounded—and the boundaries are often very irregular—by distorted and broken cells, and in sections cut from specimens preserved in alcohol, they are often found filled with a homogeneous, structureless substance. They seem to serve as reservoirs for the mucilage which abounds in these plants, and which is probably derived from the disorganization of the surrounding cells. They are most conspicuous in *Kadsura*. The xylem consists of tracheæ, fibres, and wood-parenchyma. The vessels are large, their diameter being several times greater than that of the surrounding tissue-elements, especially in *Kadsura*. The walls of the vessels are distinctly sculptured with bordered pits. The medullary rays mostly consist of a single-row of cells.

The pith consists of unlignified parenchymatous cells, among which may be found a few sclerenchymatous cells in *Schizandra* (Pl. IV, Fig. 15), and a few crystal-bearing ones in *Kadsura*.

Petiole.—Unlike those of the stem the fibro-vascular bundles of the petiole are so arranged as to present a somewhat semi-lunar form with its concavity turned to the upper side of the petiole (Pl. IV, Fig. 18). In a section cut from the upper or middle portion of the petiole, these bundles lie in a few definite groups, as is shown in the figure just referred to, (where three distinct groups are seen); but in the basal portion they are more loosely arranged. There is not present either in the phloëm or in its neighbourhood, the common form of sclerenchymatous fibre or scleroblast; but scattered about in the parenchymatous tissue surrounding the bundles there are found the crystal-bearing sclerenchymatous elements (Pl. IV, Fig. 19, *cs*). In the case of *Kadsura* a single row of parenchymatous

cells which contain a quantity of starch-grains, partly encloses the bundles and faintly represents a bundle-sheath (Pl. IV, Fig. 18, *z*; Pl. IV, Fig. 19, *z*). In *Schizandra* some parenchymatous cells lying near the bundles contain crystals of calcium oxalate. These crystals are found either singly in a cell or as an aggregate. In the former case they attain a considerable size and are octahedral or prismatic in form, but in the latter case their exact form is indeterminate. Such crystals are not met with in the stem. In the existence of mucilage-reservoirs, as well as in other points not specified here, the structure of the petiole agrees with that of the stem.

Blade.—The stomata are of the usual form. Cuticular ridges are distinctly seen on the epidermis of the midrib. The arrangement of the fibro-vascular bundles in the midrib is similar to that in the petiole. Mucilage canals are found both in the midrib and in other parts of the blade. In the latter place they generally accompany the veins sent off from the midrib. Crystal-bearing sclerenchymatous elements are found both in the midrib and in the veins of *Kadsura japonica*, while they are rarely met with in the leaf of *Schizandra nigra*. On the contrary crystals both aggregated and solitary are copiously found in the midrib and veins of *Schizandra nigra*, while they are rare in the leaf-blades of *Kadsura*. Crystals forming aggregates are also found in the epidermal cells of *Schizandra* (Pl. IV, Fig. 16).

Root.—The crystal-bearing sclerenchymatous cells, the cells containing oil-globules, and some other peculiar structures, which are found both in the stem and leaf, are also met with in the root (Pl. IV, Fig. 14, D). However, the sclerenchymatous sheath, which is constantly found in the stem, is absent even in somewhat old roots. The intercellular passages which serve as mucilage-reservoirs are found

in the roots of *K. japonica* and *S. chinensis*, but in a somewhat old root of *S. nigra*, I found the passages not yet developed, and but little mucilage present. Thus it is seen that the formation of the passages has a close relation to the production of mucilage, a fact which seems to favour the view that the mucilage is derived from the disorganization of the pre-existing tissue. As to the number of xylem-plates in the young root, *K. japonica* and *S. nigra* agree, both presenting the diarch arrangement; but in *S. chinensis* the triarch, as well as the diarch one, may be seen (Pl. IV, Fig. 17). The pith is absent in old specimens.

Tribe IV. Magnolieae.

Under this tribe I examined the following three genera:

Magnolia, Michelia, and Liriodendron.

The anatomical elements of these genera may be described together, as they present nothing characteristic enough to distinguish the genera from each other. Only one species is known to belong to *Liriodendron*, and this I examined. Four species of *Michelia* were examined, and ten species and two varieties of *Magnolia*. All of these plants are trees and generally attain a great height. Aromatic properties are prevalent among them. The names of the species examined are as follows:

Magnolia stellata, Miq.

M. parviflora, Sieb. et Zucc.

These two are ornamental trees.

M. Kobus, D. C.

Also ornamental. Its wood is used in cabinet work.

M. hypoleuca, Sieb. et Zucc.

It is one of the most useful trees, its soft wood being ex-

tensively used for various purposes. Small articles of furniture, chopping blocks, &c., are made from it. It is also used for making pencils. The charcoal obtained from it is used for polishing lacquered ware. Also the tree is ornamental.

M. salicifolia, Miq.

M. obovata, Thunb.

M. obovata, Thunb. var. (commonly known as *Kanshiu-mokuren*).

M. conspicua, Salisb.

M. conspicua, Salisb. var. *purpurescens*, Max.

M. pumila, Andr.

M. Watsoni, Hook. fil.

M. grandiflora, L.

The last seven of the above plants are ornamental exotic trees.

Michelia compressa, L.

This is an ornamental tree. Small articles of furniture are sometimes made from its wood. It is grown in the hotter parts of this country.

M. Champaca, L.

M. longifolia, Blume.

M. fuscata, Blume.

These three are ornamental exotic trees; *M. Champaca* is said to be of some medicinal value.

Liriodendron tulipifera, L.

An exotic tree.

Stem.—The epidermis is of the usual structure with the cuticle well developed in many species. Epidermal hairs are found in all species of *Michelia* and in some of *Magnolia*, (*e. g.*, *M. grandiflora*, *parvi-*

flora) ; but not in *Liriodendron*. Development of cork seems to take place in these genera immediately under the epidermis. The hypoderma is generally represented by a layer of thick-walled parenchymatous cells which are packed very closely (Pl. V., Fig. 22, *hp*). The cells constituting the hypoderma are in some cases transformed into scleroblasts, as is seen in *Magnolia grandiflora*, *hypoleuca* (Pl. V, Fig. 28, *hp*), *pumila*, *parviflora*, &c. Resin-sacs are scattered about in the external cortex. Each consists of a cell having a form similar to that of the surrounding parenchymatous cells, but distinguished from these by the nature of its contents, the thickness of its wall, and also in most cases by its size. These sacs are conspicuous in *Magnolia* and *Liriodendron*, but hardly so in *Michelia*. In many species they are found in the phloëm, and even in the pith. The contents of these sacs are probably the chief source of the aroma possessed by the species of these genera. Like that of the resin-sacs the occurrence, either singly or in groups, of scleroblasts in the external cortex is universal in these genera, These cells are, however, very rare in some species, *e. g.*, *Magnolia parviflora*. Some of them are many armed and may properly be called trichoblasts.

The sclerenchymatous fibres are greatly developed, and not only form a strong sheath to the fibro-vascular bundles, but are found mixed among the soft bast. A few scleroblasts are also found with the sclerenchymatous fibres that lie at the external limit of the phloëm, and with them constitute an almost continuous sheath. The phloëm portion consists of the usual elements. The xylem contains the tracheæ, fibrous elements, and wood-parenchyma. The vessels present bordered pits in their walls. The medullary rays vary in breadth, some consisting of three or four rows of cells, and others of a single row. Sclerenchymatous elements are generally absent in the phloëm rays.

The pith consists of unlignified cells, and is traversed in places by a kind of horizontal septa. This structure is known as "diaphragms" (Pl. V, Fig. 27, *dm*), and is universally found in the species included in the present tribe. In some species, *e.g.*, *Magnolia grandiflora*, it is very conspicuous and distinctly visible to the naked eye as streaks traversing the pith. A diaphragm consists of a horizontally stretched mass of scleroblasts, which, when highly developed, occupies an almost entire cross section of the pith, though it is inconsiderable in thickness; and this mass is continuous at several points with the wood-parenchyma that lies at the inner limit of the xylem. In *Liriodendron* the greater part of a diaphragm consists of cells with thick, pitted, but not lignified walls, while the remaining small portion is made up of scleroblasts. The diaphragms of *Magnolia hypoleuca* consist of scleroblasts having a very irregular shape. Those of *Magnolia parviflora* and *M. salicifolia* are not so well developed as those of other species.

Petiole.—The cuticle is strongly developed in *Michelia*, but not in *Liriodendron*. In *Magnolia* it is well developed in some species, but less so in others. Epidermal hairs are abundantly found in *Michelia*, but in *Liriodendron* they are almost absent. In *Magnolia* many species have the hairs well developed, while some are destitute of them. In *Magnolia parviflora* and a variety of *Magnolia obovata* (*Kaushin-mokuren*) there are found, in addition to the kind of hairs common to other species, smaller hairs which are sometimes branched (Pl. V, Fig. 24, *eh*). The hairs are generally cuticularised, but in *Magnolia Kobus*, in which a few are found, they remain cellulose. Resin-sacs occur in the outer cortex of all the species, and even in the phloëm of some. The scleroblasts, which are often many-armed, are generally found in the external cortex; but in the single case of *Liriodendron* they are almost absent. The number of fibro-vascular bundles which enter

the petiole is variable, in some species exceeding twenty, in others being much fewer. These bundles are generally isolated from one another in the basal portion of the petiole, but in its upper portion they become gradually united so as to form a ring enclosing the pith. This arrangement is usually very regular, but sometimes several bundles stand isolated outside the main ring, and disturb the regularity of the arrangement. The sclerenchymatous fibres are very well developed and constitute the bundle sheath. They are generally unlignified at the basal portion of the petiole, but in *Liriodendron* some of these fibres are lignified even in the basal portion. The diaphragms are present in the species examined. In several cases, however, they are only faintly represented by a few scleroblasts found in the pith; while in others (*e. g.*, *Michelia fuscata*), a broad mass of sclerenchymatous cells constitutes the diaphragm.

Blade.—With the exception of *Liriodendron* the cuticle is generally well-developed. In *Michelia* and *Magnolia* epidermal hairs are generally met with, especially on the lower surface of the leaf; while in *Liriodendron* many epidermal cells of the lower side of the leaf are provided each with a little protuberance, which is really an imperfect hair. In many cases the hair is not an elongation from a single epidermal cell, but several cells take part in forming its basal portion, as will be seen in referring to the figures, 25 and 26 (Pl. V). The stomata are generally elliptical in form, but in *Magnolia grandiflora* they are rather roundish, the curvature of the guard-cells being very great. In *Magnolia pumila* the cuticle is much raised at the entrance of the stomata, and presents a cup-like appearance, reminding us of the stomata of *Trochodendron* (Pl. V, Fig. 30). In *Magnolia grandiflora* there is found between the epidermis and the palisade-parenchyma a single layer of parenchymatous cells, which do not contain chlorophyll grains (Pl. V, Fig. 29). This layer extends to the very margin of

the leaf-blade, and some of the cells constituting it become sclerotic as this region is approached, and, together with the sclerenchymatous fibres that abound there, form a very strong margin to the blade. The blade of *Magnolia grandiflora* is also itself very thick, the palisade-parenchyma consisting of several layers of cells, but in *Magnolia salicifolia*, which has very thin leaves, there is only a single layer of cells constituting the palisade parenchyma. Although there are the few just noticed peculiarities to be detected in certain species, the general structure of the blade is similar in the three genera of *Magnolieæ*. The fibro-vascular bundles are circularly arranged in the midrib, as they also are in the petiole. Resin-sacs are found in the leaf-blades of all the species examined. The occurrence of stone-cells is also very general. *Magnolia pumila* contains somewhat elongated sclerenchymatous cells in the hypodermal region of the lower side of the midrib. Red patches are often met with in the mesophyll of some species of *Magnolia*, *e. g.*, *M. obovata*, *M. conspicua*, and *M. conspicua* var. *purpurescens*. These were proved to be tannin.

Root.—The general structure of the older root does not much deviate from that of the stem. Resin-sacs are present without exception in the outer cortex. This region also contains a number of scleroblasts in some species, while it is destitute of them in others. The sclerenchymatous fibres are generally developed at the external limit of the phloëm. Pith is absent, and its place is occupied by a group of elongated woody fibres. In the young root the number of xylem-plates is very variable. Thus in *Liriodendron* their arrangement is diarch or triarch, but mostly the former. In *Michelia* it is triarch or tetrarch (Pl. V, Fig. 32, A). In *Magnolia* diarch, triarch, and tetrarch arrangements are each common; but in one species, namely, *Magnolia grandiflora*, the arrangement varies from the tetrarch to the heptarch. Generally, when the root becomes a little older the xylem-plates which

are separate at first, tend to unite and form one body of xylem; but in *Magnolia grandiflora* they continue separate for a comparatively long time, so that there remains in the central portion a mass of unlignified cells. In *Kanshin-mokuren*, (a variety of *Magnolia oborata*), I saw only a continuous mass of lignified cells instead of separate xylem-plates, even in the youngest specimens obtainable. In general, the endodermis is distinctly marked off from the surrounding tissues (Pl. V, Fig. 32, *e*), and in most cases it is traceable even in somewhat old roots. In *Michelia compressa* and *Magnolia conspicua*, var. *purpurascens*, somewhat elongated cells with lignified walls appear very early in the cortex (Pl. V, Fig. 32, *cs*), while the xylem-plates still remain separate. The walls of these cells in the latter species are not uniform in thickness, the part of the wall facing the circumference being not much lignified, while that facing the centre is thickened and lignified.

The cortex of the older root of *Magnolia Kobus* contains many elongated peculiar cells, found near the intercellular spaces of schizogenetic origin, which also abound in the root of this plant.

When the roots of some species of *Magnolieæ*, such as *Michelia Champaca*, *Mich. compressa*, *Magnolia Kobus*, &c., are kept in alcohol they impart a red colour to it.

Anatomical Characters of Different Genera Compared.

In the foregoing pages the anatomical characters of each genus of the present family having been described, it is now possible to give a comparative view of them. The structural points to be compared will be treated under the same four principal heads as used in describing each genus: Stem, Petiole, Blade, and Root.

Stem.

The characters of the stem require fuller treatment than the rest, and will be compared under the several structures.

1. *Epidermis.*—The epidermis generally consists of a single layer

of cells. The cuticle is sometimes well developed, as in *Illicium*, *Drimys*, *Trochodendron*, and many species of *Magnolia*. Epidermal hairs are found only in the two genera, *Michelia* and *Magnolia* and in the latter genus are almost absent in some species.

2. *Cork*.—This structure presents no peculiarity. As with many other Dicotyledons, cork is here developed immediately beneath the epidermis. In *Euptelæa* it is conspicuous from the size of its cells.

3. *Hypoderma*.—This is totally wanting in *Illicium*, *Drimys*, *Kadzura*, and *Schizandra* but present in all the other genera. When present, it consists of a layer of thick-walled and closely packed parenchymatous cells (Pl. V, Fig. 22, *hp*), and in some species of *Magnolia*, as *M. grandiflora*, *hypoleuca* (Pl. V, Fig. 28, *hp*), &c., these cells are transformed into scleroblasts.

4. *Secretory Reservoirs*.—Two kinds of these are found, resin- or oil-sacs and crystal-containing sacs, the latter chiefly found in *Euptelæa* (Pl. II, Fig. 2) and *Cercidiphyllum* (Pl. II, Fig. 5). The crystals are of calcium oxalate in both genera; but in *Euptelæa* they exist in the sac as an aggregate and are confined to the outer cortex, while in *Cercidiphyllum* they are not aggregated, and the sacs containing them exist in the phloëm, as well as in the outer cortex. In *Schizandra* crystal-containing sacs are confined to the leaf (petiole and blade). Resin-sacs universally occur in the external cortex of *Magnoliæ*, and in many cases are found in the phloëm, and sometimes even in the pith. Cells containing oily substances are also met with in the external cortex of *Schizandra* (Pl. IV, Fig. 14, C.D). Resin-sacs are also present in *Drimys* and *Illicium Tashiroi*, but absent in *Illicium religiosum*. *Trochodendron* is entirely destitute of any kind of reservoirs.

5. *Scleroblasts and Trichoblasts (in the external cortex)*.—The latter are simply a modified form of the former. Both forms of cells are

met with in Magnoliaceæ. Trichoblasts are very abundantly found in the external cortex of *Trochodendron*, and send their arms freely into the intercellular spaces which also abound in this region (Pl. III, Figs. 7 and 9). Their presence is a feature that distinguishes this genus from others. Scleroblasts are generally found in *Magnolieæ*, and in many cases are accompanied by trichoblasts. The external cortex of *Illicium*, (perhaps of *Drimys* also), *Euptelea*, and *Cercidiphyllum* is entirely destitute of them. *Kadzura* and *Schizandra* contain in their outer cortex and less often in their inner one a number of crystal-bearing sclerenchymatous cells (Pl. IV, Fig. 20), which are nothing more than scleroblasts with numerous crystals imbedded in their walls. Some of these cells are elongated and form crystal-bearing sclerenchymatous fibres.

6. *Fibro-vascular Bundles.*—The fibro-vascular bundles existing in the stem of Magnoliaceæ belong to the so-called collateral bundles, which are universally found in the stem of phanerogams, so that it is unnecessary to give an account of their general features. But there are certain structural points in the bundles peculiar to different genera which need to be noticed briefly under the following heads:

(1) *Sclerenchymatous Sheath.*—This is made up of sclerenchymatous fibres often accompanied by scleroblasts, and is more or less developed in all the genera. However, in *Illicium* it is very imperfectly represented by scattered fibres that are found at the external limit of the phloëm. It is poorly developed in *Kadzura* and *Shizandra* (Pl. IV, Fig. 13, *sl*); rather well developed in *Euptelea* and *Cercidiphyllum*; in *Trochodendron* and all the three genera of *Magnolieæ* most strongly developed and, together with a number of scleroblasts existing there, constitutes an almost continuous ring of sclerenchyma. Such is also the case in *Drimys*.

(2) *Phloëm.*—The soft portion of the phloëm probably contains

those elements which are usually found in this region of other Dicotyledons, but clear detection of them was generally very difficult, and the presence of sieve-tubes was proved only in a few species. In *Cercidiphyllum* and all the three genera of *Magnolieæ* the hard-bast or sclerenchymatous fibres are especially well developed, and are found among the soft bast. Also in *Kadzura* and *Schizandra* sclerenchymatous fibres are found among the soft bast, but these are heavily loaded with minute crystals, and transformed into the crystal-bearing sclerenchymatous fibres (Pl. IV. Fig. 20). Sacs containing crystals exist in the phloëm, as well as in the external cortex of *Cercidiphyllum* (Pl. II, Fig. 5). In *Kadzura* and *Schizandra* it is a remarkable feature that there exist in this region large intercellular passages, which serve as the reservoirs of mucilage.

(3) *Xylem.*—This portion generally consists of tracheæ, fibrous elements, and wood-parenchyma. But *Trochodendron* and *Drimys* are exceptional in this matter, their wood consisting almost exclusively of tracheïds. This peculiar structure of the xylem distinguishes these two genera from others of the Magnoliaceæ, and at the same time allies them with Coniferæ. But, as stated by Prantl,* their wood may be distinguished from that of the latter family by the fact that in Coniferæ the cells of the medullary rays have their longer axis horizontal, while in *Drimys* and *Trochodendron* most of the cells of the medullary rays have their longer axis vertical. Bordered pits are found in the tracheïds of *Drimys* and *Trochodendron*; in the other genera where vessels are present, pits of this kind are generally found in their walls. It may be added that the exact determination of the various elements that constitute the zylem was not very easy.

(4) *Medullary Rays.*—Their breadth varies, consisting mostly of a single row of cells in *Illicium*, *Cercidiphyllum*, *Schizandra*, and *Kadzura*,

* Engler and Prantl, *Die natürlichen Pflanzenfamilien.*

but of several rows in *Trochodendron*. In *Magnolieæ* different breadths are found. In the phloëm rays of *Euptelea* a few scattered scleroblasts find their way among soft parenchymatous cells (Pl. II, Fig. 1, *s*).

7. *Pith*.—The parenchymatous cells of the pith are completely lignified in *Cercidiphyllum*, *Drimys*, and *Trochodendron*, even in young specimens. This is also the case in *Illicium Tashiroi*, but in *Illicium religiosum* they are unlignified in young specimens. The young stem of *Euptelea* often has a small portion of the pith remaining unlignified. The parenchymatous cells of the pith are unlignified in *Kadzura* and *Schizandra*, but a few sclerenchymatous cells are found in it (Pl. IV, Fig. 15). *Magnolieæ* are distinguished by the existence of diaphragms in their pith (Pl. V, Fig. 27, *dm*), the only part of the pith that here becomes lignified. As I have stated before, the asserted existence of stone-cells in the pith of *Illicieæ* is not verified by my specimens.

Petiole.

The structure of the stem having been comparatively described, there remains scarcely anything to be said of the petiole. I will here only notice a few anatomical points in which the structure of the petiole deviates from that of the stem.

The epidermal hairs are generally more strongly developed in the petiole than in the stem, as we see in many species of *Magnolia*; and even *Euptelea*, which bears no hair on its stem, has a few on the petiole. In those species which contain resin-sacs, scleroblasts, and trichoblasts in their stem, the presence of the same in the petiole is almost certain, as is also the case with crystal-bearing sclerenchymatous elements. A number of crystals is found in the outer cortex of the petiole of *Schizandra*, though none is present in the stem. The fibro-

vascular bundles are in many cases arranged as in the stem ; that is, in the cross section of the petiole, they present a circular arrangement enclosing the pith in the centre (Pl. II. Fig. 6). But in the petiole of *Illicium*, *Drimys*, *Trochodendron*, *Kadzura*, and *Schizandra*, the fibro-vascular bundles have a semi-circular arrangement with the concavity turned to the upper side of the petiole (Pl. III, Fig. 7). The sclerenchymatous sheath of the bundles is often wanting, as in the petiole of *Illicium*, *Drimys*, and *Schizandra*. *Trochodendron* has the sheath consisting of unlignified fibres, while all the other genera have well developed sclerenchymatous sheaths (Pl. II, Fig. 6). The fibres forming these sheaths remain unlignified at the basal portion of the petiole in all cases, except in the petiole of *Liriodendron*, where a few fibres are found lignified. Whether this local non-lignification of the fibres has some physiological meaning or not still remains undecided. In *Magnolieæ* diaphragms are found in the pith-portion of the petiole as in that of the stem.

Blade.

The general structure of the blade does not deviate from the dicotyledonous type. The upper portion of it is made up of palisade-parenchyma, which consists of vertically elongated cells varying in different species from one to several layers. Where these cells lie directly over the main ribs, their height is generally diminished and their form becomes almost round. The lower portion of the blade is generally made up of loose parenchyma ; but in those parts lying directly under the main ribs there is found a more compact parenchymatous tissue. In some genera, namely, *Illicium*, *Trochodendron*, *Magnolia*, and *Michelia*, the blade has a well developed cuticle. In *Schizandreæ* the cuticle of the midrib—especially that of the lower surface—forms fine ridges running longitudinally (Pl. II, Fig. 3).

These ridges are also observable in *Euptelea*, and less distinctly in *Cercidiphyllum*, but are wanting in *Trochodendron* which with these two genera forms *Trochodendreæ*. Epidermal hairs are generally found in *Magnolia* and *Michelia*, and also in *Euptelea*, but in other genera are almost wanting. In general the stomata do not present any peculiarity, being elliptical in form and accompanied by the two guard-cells, but in *Magnolia grandiflora* their form is nearly spherical and the guard-cells are pressed outward, so that they rest upon the neighbouring epidermal cells. Both in *Magnolia pumila* and *Trochodendron aralioides*, the cuticle is much raised around the entrance of the stomata, and presents a cup-like appearance (Pl. III, Fig. 10; Pl. V, Fig. 30). The arrangement of the fibro-vascular bundles of the midrib is circular in *Magnolieæ*, semi-circular in *Illicieæ* and *Schizandreæ*. In all these cases the bundle arrangement of the petiole agrees with that of the midrib. In *Cercidiphyllum* and *Euptelea* the bundles are circularly arranged in the petiole, but in the former they are semi-circularly arranged in the midrib, while in the latter their circle is incomplete in this part. But *Trochodendron aralioides* has the bundles semi-circularly arranged both in petiole and midrib. Thus in this respect the three genera of *Trochodendreæ* do not agree with one another. Resin-sacs are present in the mesophyll, as well as in the midrib of the examined three genera of *Magnolieæ*. They are also present in these parts of *Drimys dipetala*, but have not been observed in those of *Illicium religiosum* or of *Schizandreæ* and *Trochodendreæ*, although in the former of these the occurrence of mucilage canals both in the mesophyll and the midrib is very common. The blade in a few genera contains crystals. Thus aggregates of crystals are found in the cortical region of the midrib of *Euptelea* and *Cercidiphyllum*, and in that of *Schizandra*. In *Drimys dipetala* the epidermal cells contain a number of crystals. As to the existence of scleroblasts, trichoblasts,

and crystal-bearing sclerenchymatous elements, what I have stated in the case of the stem is applicable here to the blade.

Root.

The root, when it becomes a little older, presents the same structure as the stem, except that the central portion, which corresponds to the pith of the stem, is occupied by the xylem. Generally when scleroblasts, secretory reservoirs, &c. are found in the stem, they are also present in the root; but in the case of *Illicium religiosum*, sclerenchymatous elements are totally absent in the root, though a few fibres are present in the stem. In the young root radial bundles with the endodermis are present in the central portion, and are surrounded by the cortical parenchyma. The cells constituting the endodermis generally remain cellulose. The Caspary point is not clearly seen. Both endodermis and pericambium are generally well marked in the young root, except in that of *Trochodendron* (Pl. III, Fig. 8). In a few cases a number of cells with lignified thick walls are found in the cortical portion (Pl. III, Fig. 8, *cs*; Pl. V, Fig. 32, *cs*). The arrangement of the xylem-plates varies, there being usually found the diarch, triarch, and tetrarch types, while the number is not constant even in the same individual. In the single case of *Magnolia grandiflora* the tetrarch to heptarch arrangement is found. In this respect, therefore, any definite statement can not be given.

In all cases the number of xylem-plates was determined only in young branchlets of the secondary root, and not in the main root proceeding direct from the seedling, of which, as I have already mentioned, I could obtain no specimens.

Concluding Remarks.

Having given a comparative view of the anatomical characters of different genera of the present family, it remains now to summarise the results of my observations, which are mainly negative. Thus, I

have been unable to find any anatomical character that might serve to distinguish Magnoliaceae as a whole from other dicotyledonous families. There is certainly not a single character common to all the members of this family, except such as are common to them and the members of other dicotyledonous families. Again, I have been unable to find in most of the species of this family any anatomical character that might serve to distinguish them one from another. However, I found that each of the distinct groups included in the family is marked out by certain anatomical characters, as the following synopsis shows.

A. With diaphragms in the pith:

Magnolia, *Michelia*, *Liriodendron*. } Tribe *Magnolieæ*.

B. Without diaphragms in the pith:

I. With crystal-bearing sclerenchymatous elements in the cortical portion of both stem and petiole:

Kadsura, *Schizandra*. } Tribe *Schizandreæ*.

II. Without crystal-bearing sclerenchymatous elements in the cortical portion of stem and petiole:

1. A few scattered sclerenchymatous fibres present in the cortex of the stem; though totally absent in that of the petiole, and that of the midrib of the blade. Resin-sacs either indistinct or wanting:

Illicium. Tribe *Illicieæ*.

2. Sclerenchymatous sheath well developed, and resin-sacs present both in stem and leaf:

Drimys. Tribe *Illicieæ*.

3. Sclerenchymatous fibres well developed in the

cortical portion of both stem and petiole, (in a single case only the fibres in the petiole remaining unlignified). Resin-sacs wanting:

a. With a great number of trichoblasts in the outer cortex, as well as in the mesophyll; wood almost exclusively consisting of tracheïds:

Trochodendron. Tribe *Trochodendreæ.*

b. Cells containing single prismatic crystals abundantly found in the cortex of the stem:

Cercidiphyllum. Tribe *Trochodendreæ.*

c. Cells containing an aggregate of minute crystals often present in the outer cortex of the stem; each aggregate presenting a stellate appearance:

Euptelea. Tribe *Trochodendreæ.*

As will be seen from the above synopsis, the four tribes of Magnoliaceæ may be distinguished from one another by certain anatomical characters more or less peculiar to each. Of these four tribes *Magnolieæ* and *Schizandreæ* are well defined in their characters and may be easily distinguished not only from each other, but from either of the other tribes. Although these two tribes are included in the same family by most botanists, yet there does not exist any anatomical character which seems to be sufficient to connect them. It is true that the secretory reservoirs found in the cortex of *Magnolieæ*, are faintly represented in *Schizandreæ* by reservoirs quite different in form; and that the diaphragms in the pith of *Magnolieæ* are said to be represented in *Schizandreæ* by a few sclerenchymatous cells.* But I hesitate to consider these points as sufficient connecting links,

* Baillon, *Natural History of Plants* (Translated from the French).

since there exist in other anatomical characters of these tribes such marked differences as those which I have pointed out. Thus, then from a purely anatomical point of view, *Magnolieæ* and *Schizandreæ* may be taken without any impropriety to be two distinct groups.

In *Illicieæ* the only two genera, *Illicium* and *Drimys*, which I examined, are markedly different in some points, though similar in general structure. On the one hand, the semicircular arrangement of the fibro-vascular bundles of the petiole allies this tribe to *Schizandreæ*, and on the other hand, the existence of resin-sacs in *Drimys*, and probably in some species of *Illicium*, allies it to *Magnolieæ*. Still the tribe *Illicieæ* is widely separated from these two tribes by the absence of certain anatomical characters possessed by one or the other of them: for instance, both the mucilage-canals and crystal-bearing sclerenchymatous elements of the *Schizandreæ*, and the diaphragms and scleroblasts which are found respectively in the pith and in the outer cortex of *Magnolieæ*, are wanting. Thus, here again is a tribe which may be anatomically considered as a distinct group.

The wood in *Drimys* is almost exclusively made up of tracheïds, and the leaf-petiole has its fibro-vascular bundles arranged in a semicircle, points which ally this genus to *Trochodendron* in another tribe. Thus *Illicieæ* and *Trochodendreæ* might seem to be connected through these two genera, were it not that the very characters that ally *Trochodendron* to *Illicieæ*, widely separate this genus from *Euptelea* and *Cercidiphyllum*, the other two genera of *Trochodendreæ*. Further, these other two genera are not only but slightly related to each other, but are also destitute of any anatomical characters that may serve to connect them with the members of the other tribes. Such being the case, it is evident that the tribe *Trochodendreæ* is very indefinite in its anatomical characters, and that on these grounds we ought to place *Trochodendron* in the *Illicieæ*, and establish a new tribe, *Eupteleæ*,

which would include *Euptelea* and *Cercidiphyllum*, the remaining two genera of *Trochodendreæ*.

Summing up, the *Magnoliaceæ* may be split up into the following four groups:

First group, identical with *Magnolieæ*.

Second group, identical with *Schizandreæ*.

Third group, consisting of *Trochodendron* and the genera of *Illicieæ*.

Fourth group, consisting of *Euptelea* and *Cercidiphyllum*.

Now, it is worth noticing that the four groups into which Magnoliaceæ might be divided as the result of a purely anatomical study, correspond in the main to its four tribes, the distinction between which is based on the external characters; and we should not perhaps be wrong in making the general statement that resemblance in the external characters of certain plants indicates that there also exists resemblance in their internal or anatomical structure.

As we have just seen *Illicium* and *Drimys* present some marked differences in their anatomical characters, as do also to some extent *Euptelea*, *Cercidiphyllum*, and *Trochodendron*, which are the three genera constituting *Trochodendreæ*; but *Magnolia*, *Michelia*, and *Liriodendron*, the three genera belonging to *Magnolieæ* are so similar in their internal structure as not to be distinguishable from one another, and this is also the case with the two genera of *Schizandreæ*, namely, *Schizandra* and *Kadsura*. From this we see that anatomical structure is not always to be relied upon in distinguishing different genera of the same tribe or family. However, we must remember that *Schizandra* and *Kadsura* have differences so slight in their generic characters* that Baillon combines them into one genus.† Again, *Magnolia*, *Michelia*, and *Liriodendron*

* Generic characters are generally founded on external peculiarities, and not on internal ones.

† Baillon, *The Natural History of Plants*, (Translated from the French).

are so related in their external characters that there are not wanting cases in which some species of one of the three genera is often placed in either of the other two.‡ If we except the case of such closely related genera, it is probably the rule that any two genera of the same tribe or family present anatomical characters of such difference as to distinguish them from each other.

If it is difficult in some cases to distinguish certain genera of the same tribe or family from one another by anatomical characters alone, much more would it be so to distinguish in this way species included in the same genus. Such is actually the case in the present family, and so it would perhaps be in other families.

I wish to express here my obligations to Professor Edward Divers for his kindness in looking through the present article and in suggesting many improvements in English.

‡ Thus, *Magnolia pumila*, Andr., *Liriodendron coco*, Lour. and *Liriodendron lilifera*, Linn, are synonymous; also *Michelia fuscata*, Blume, *Magnolia fuscata*, Andr., and *Liriodendron Figo*, D. C., are synonymous—*The Journal of the Linnean Society*, Vol. XXIII, No. 150.

Explanation of Plates.

List of Reference Letters.

(Those not found here are explained where they occur.)

ac. aggregate of crystals.
c. cambial zone.
cp. cortical parenchyma.
cr. crystals.
cs. crystal-bearing sclerenchymatous element.
cu. cuticle.
dm. diaphragm.
e. endodermis.
eh. epidermal hair.
ep. epidermis.
es. elongated sclerenchymatous cells.
g. guard cells.
hp. hypoderma.
i. intercellular space.
il. inner limit of xylem.
k. cork.
m. cells constituting medullary ray.
md. pith.
mp. passage containing mucilage.
mp'. passage containing mucilage in process of formation.
p. parenchymatous cells.
pc. pericambium.
ph. phloem.
pm. cell constituting phloem ray.
rc. sac or reservoir containing crystals.
ro. sac or reservoir containing oily or resinous substance.
s. stone-cells or scleroblasts.

sg. group of sclerenchymatous fibres.

sl. sclerenchymatous fibres or bast fibres.

ss. sclerenchymatous sheath.

t. trichoblasts.

x. xylem.

z. one of the cells that form a row representing a bundle-sheath.

PLATE II.

Plate II.

FIG. 1.—Cross section of phloëm and cambial zone of *Euptelea polyandra*. ×285. *w*. a zone of newly formed wood, where lignification is incomplete.

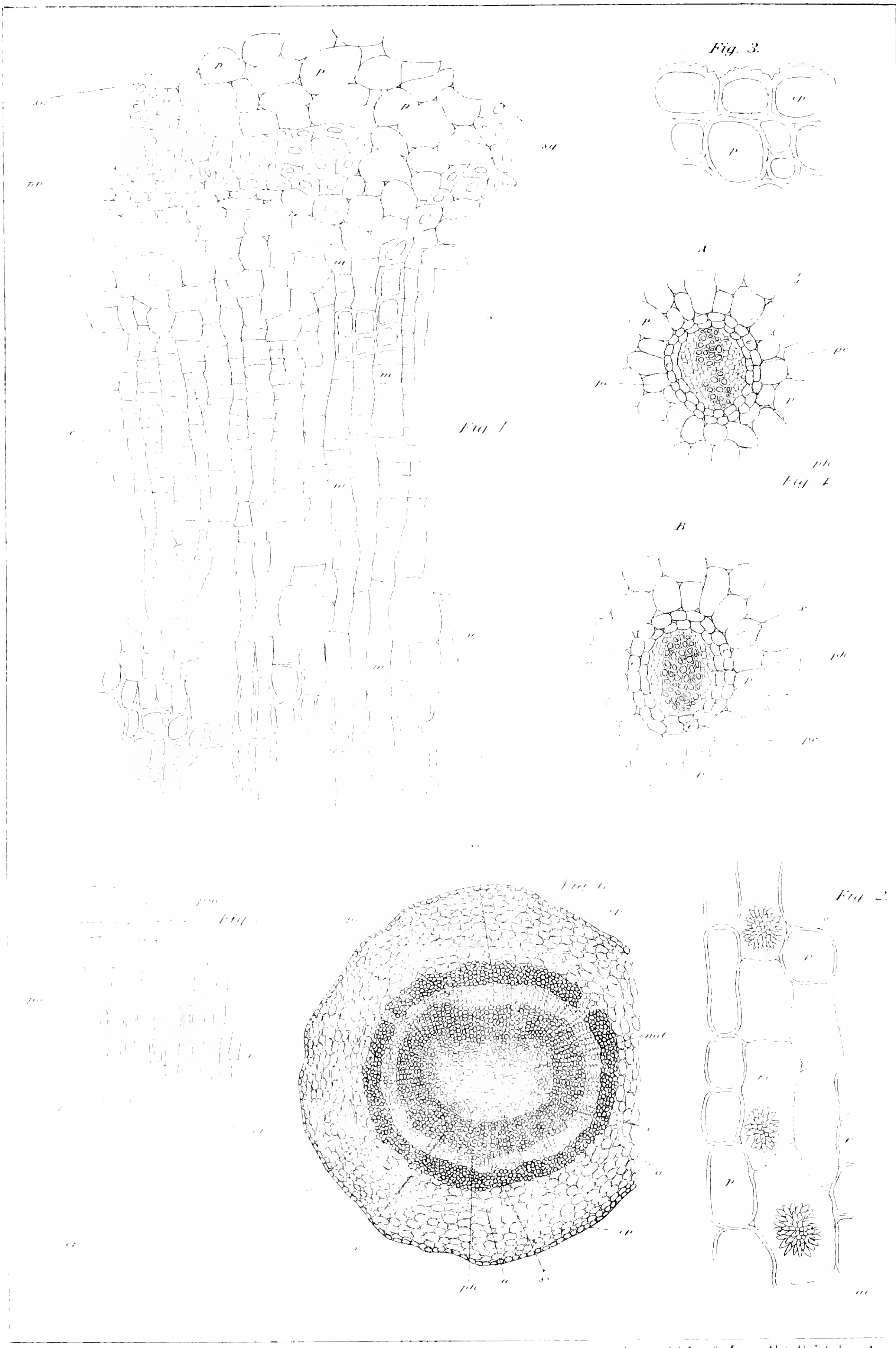
FIG. 2.—Longitudinal radial section of outer cortex of *Euptelea polyandra*, showing three crystal-containing sacs. ×355.

FIG. 3.—Cross section of epidermis of midrib of leaf of *Euptelea polyandra*, presenting cuticular ridges in cross section. ×355.

FIG. 4.—Cross section of young root of *Euptelea polyandra*: A, younger; B, a little older stage. ×200.

FIG. 5.—Longitudinal radial section of a portion of phloëm of *Cercidiphyllum japonicum*, showing crystal-containing sacs; *l* indicates the side nearer to the centre of the stem; *r*, the side farther from the centre. ×200.

FIG. 6.—Cross section of the middle portion of petiole of *Cercidiphyllum japonicum*: *a*, *a*, *a*, mark the zone where distorted cells are found. ×45.



S. Matsuda, del.

Lith. & Imp. the Seishibunsha.

Plate III.

FIG. 7.—Cross section of the middle portion of petiole of *Trochodendron aralioides.* ×45.

FIG. 8.—Cross section of young root of *Trochodendron aralioides;* endodermis and pericambium not well marked. ×200.

FIG. 9.—A trichoblast (separated by maceration) from outer cortex of *Trochodendron aralioides.* ×130.

FIG. 10.—Cross section of a stoma of *Trochodendron aralioides; cs*, cup-shaped cuticular elevation at the entrance of the stoma. ×445.

FIG. 11.—Cross section of phloem and a portion of xylem of *Illicium religiosum.* ×285.

FIG. 12.—A, cross section of a small portion of outer cortex of *Illicium Tashiroi*, showing a resin sac; B, that of *Drimys dipetala;* (only dried specimens were examined). ×200.

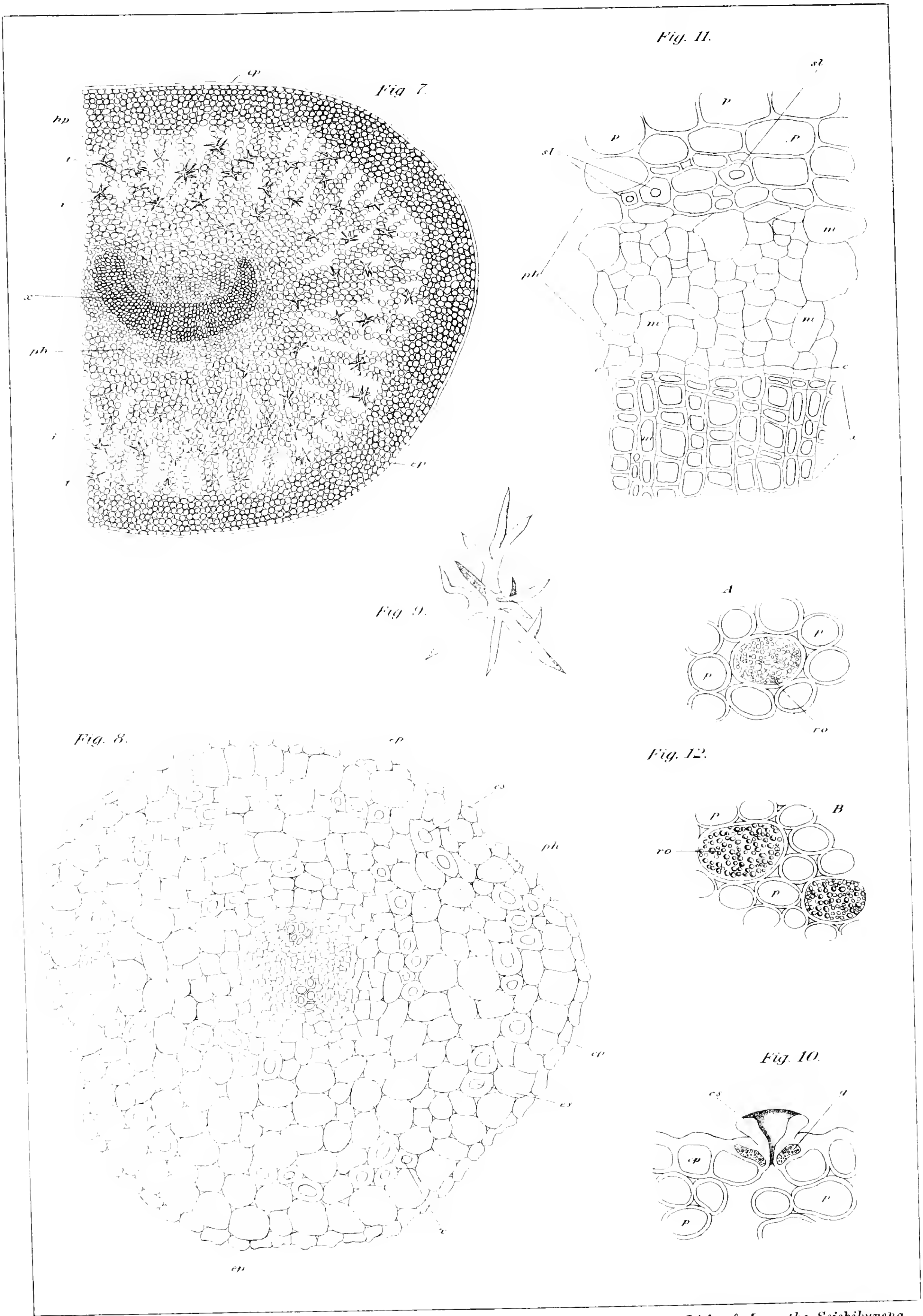


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PLATE IV.

Plate IV.

FIG. 13.—Cross section of outer cortex of *Kadzura japonica*. ×110.

FIG. 14.—A, secretory reservoir in longitudinal tangential section of outer cortex of *Schizandra nigra*; B, that of *S. chinensis*; C, that of *K. japonica*; D, that of root of *K. japonica*. All ×130.

FIG. 15.—Longitudinal section of pith of *Schizandra nigra*. ×45.

FIG. 16.—Superficial view of an epidermal cell of *Schizandra nigra* containing an aggregate of crystals. ×445.

FIG. 17.—A, cross section of young root of *S. chinensis*, presenting the triarch arrangement of xylem-plates; B, that of a little older one of the same. ×200.

FIG. 18.—Cross section of the middle portion of petiole of *Kadzura japonica*. ×45.

FIG. 19.—Cross section of a part of the fibro-vascular bundles of petiole of *Kadzura japonica*. ×200.

FIG. 20.—Various forms of crystal-bearing sclerenchymatous elements; A and B, from stem of *K. japonica*; C and D, from its petiole. ×200.

FIG. 21.—Septate bast-fibre from *K. japonica*. ×70.

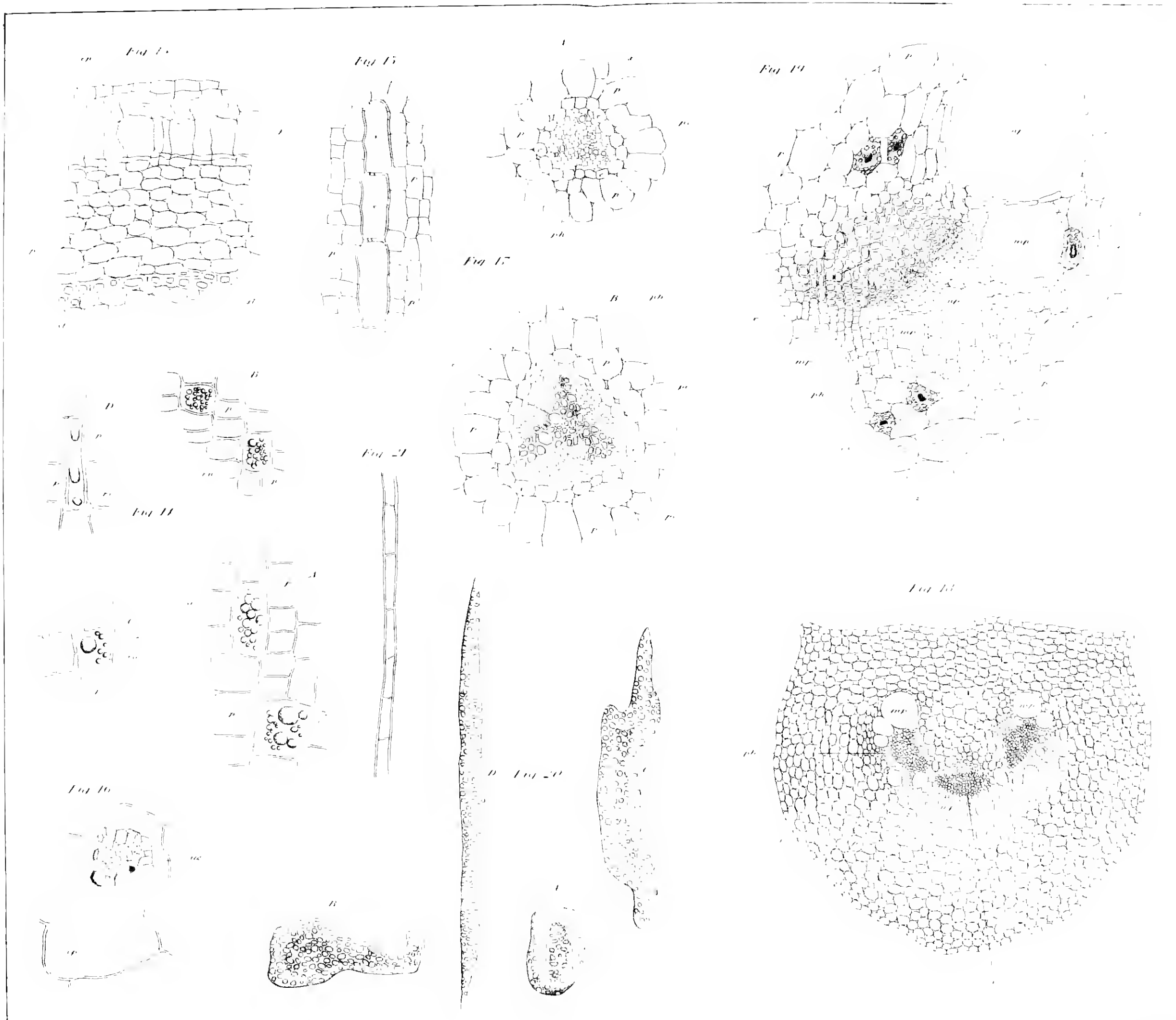

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PLATE V.

Plate V.

Fig. 22.—Cross section of outer cortex of *Magnolia oborata*. ×130.

Fig. 23.—Longitudinal tangential section of a small portion of outer cortex of *Michelia Champaca* showing resin sacs. ×130.

Fig. 24.—Epidermal hairs of the smaller kind found in *Magnolia oborata*, var. *(Kanshu-mokuren)*. ×355.

Fig. 25.—The basal portion of an epidermal hair from the lower side of the leaf of *Magnolia conspicua*, var. *purpurescens*. ×355.

Fig. 26.—Superficial view of a portion of epidermis of *Magnolia Watsoni*: *bs*, basal portion of an epidermal hair cut crosswise. ×355.

Fig. 27.—A, pith of *Michelia compressa* in longitudinal section showing "diaphragms," (×15): B, the same in cross section, and more magnified (×70); C, in longitudinal section, and magnified as in B.

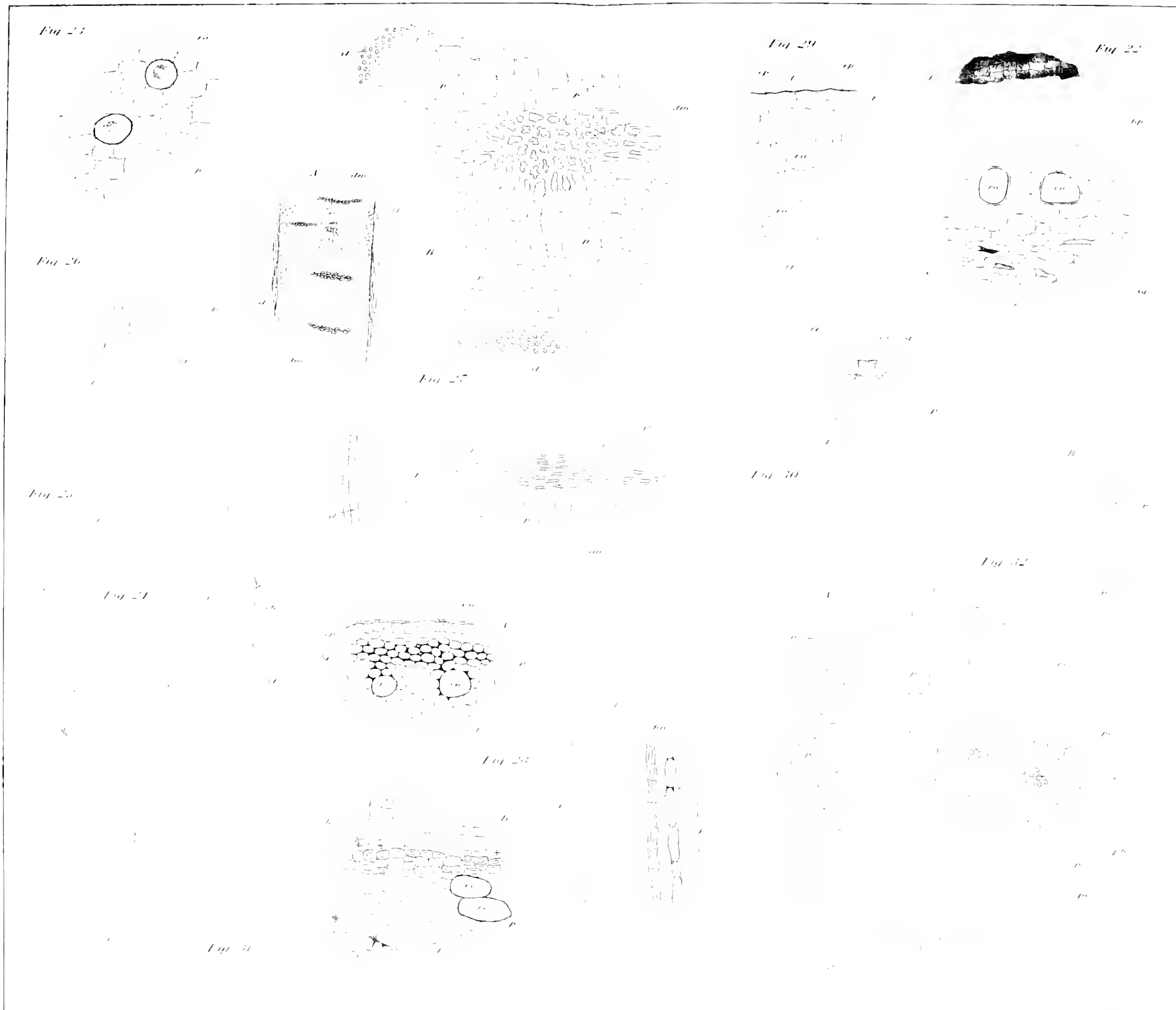
Fig. 28.—Outer cortex of *Magnolia hypoleuca*, showing hypoderma; A, cross section of younger stage; B, cross section of a little older stage; C, longitudinal radial section of the same stage as B. ×130.

Fig. 29.—Cross section of the leaf of *Magnolia grandiflora*. *l*, a layer of cells lying directly below the epidermis of the upper side of the leaf. ×130.

Fig. 30.—Stoma of *Magnolia pumila* in cross section: *cs*, a cup-shaped cuticular elevation at the entrance of the stoma. ×445.

Fig. 31.—Stoma of *Magnolia grandiflora* in crosss ection; guard cells lying directly upon epidermal cells. ×355.

Fig. 32.—A, cross section of young root of *Magnolia compressa*; B, longitudinal section of a small portion of the same, showing an elongated sclerenchymatous cell. ×200.



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Researches on the Multiplication of Elliptic Functions.

By

R. Fujisawa.

Jacobi, in one of the *Suites des notices sur les fonctions elliptiques*, gives, without demonstration, remarkable expressions of $\operatorname{sn} 2u$, $\operatorname{sn} 3u$, $\operatorname{sn} 4u$ and $\operatorname{sn} 5u$ in terms of the differential coefficients of $\sqrt{x^2(1-x^2)(1-k^2x^2)}$ and $\sqrt{\dfrac{(1-x^2)(1-k^2x^2)}{x^2}}$ taken with respect to x^2, whereby x stands for $\operatorname{sn} u$.* Namely, writing

$$\sqrt{x^2(1-x^2)(1-k^2x^2)} = A, \qquad \sqrt{\frac{(1-x^2)(1-k^2x^2)}{x^2}} = B,$$

we have

$$\operatorname{sn} 2u = -\frac{1}{x^2}\,\frac{1}{\dfrac{dB}{d(x^2)}},$$

$$\operatorname{sn} 3u = -x^3\,\frac{\dfrac{d^2B}{d(x^2)^2}}{\dfrac{d^2A}{d(x^2)^2}},$$

$$\operatorname{sn} 4u = -\frac{1}{x^4}\,\frac{\dfrac{1}{2.3}\,\dfrac{d^3A}{d(x^2)^3}}{\dfrac{1}{2}\,\dfrac{d^2B}{d(x^2)^2}\;\dfrac{1}{2}\,\dfrac{d^2B}{d(x^2)^2}-\dfrac{dB}{d(x^2)}\;\dfrac{1}{2.3}\,\dfrac{d^3B}{d(x^2)^3}},$$

$$\operatorname{sn} 5u = x^5\,\frac{\dfrac{1}{2.3}\,\dfrac{d^3B}{d(x^2)^3}\;\dfrac{1}{2.3}\,\dfrac{d^3B}{d(x^2)^3}-\dfrac{1}{2}\,\dfrac{d^2B}{d(x^2)^2}\;\dfrac{1}{2.3.4}\,\dfrac{d^4B}{d(x^2)^4}}{\dfrac{1}{2.3}\,\dfrac{d^3A}{d(x^2)^3}\;\dfrac{1}{2.3}\,\dfrac{d^3A}{d(x^2)^3}-\dfrac{1}{2}\,\dfrac{d^2A}{d(x^2)^2}\;\dfrac{1}{2.3.4}\,\dfrac{d^4A}{d(x^2)^4}}.$$

* Crelle's Journal, Bd. IV, pp. 185-193, and Jacobi's gesammelte Werke, Bd. I, pp. 266-275.

Jacobi adds "*la loi général de la composition de ces expressions est aisée à saisir,*" and further remarks that we shall have analogous formulæ by using, instead of the differential coefficients of A and B, those of

$$\frac{1}{\sqrt{x^2(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad \frac{1}{\sqrt{\dfrac{(1-x^2)(1-k^2x^2)}{x^2}}}.$$

The general form of these expressions is, however, by no means easy to infer from the particular cases just given, and I have tried to trace among the writings of Jacobi, the steps which might have led him to these expressions, but without success.

The present memoir is divided into two parts. In the first part, the multiplication-formulæ of elliptic functions are derived from Abel's theorem for the elliptic integral of the first kind. It will be seen that one of the results arrived at is the general formula in question. It appears, however, highly improbable that Jacobi obtained his formulæ in this way.

In the paper just alluded to, Jacobi gives, also without demonstration, the partial differential equation satisfied by the numerators and denominator of the multiplication-formulæ. This partial differential equation has since been obtained by Betti,* Cayley,** Briot et Bouquet,† and others; but the final results to be obtained by applying it to the actual evaluation of the numerical constants involved in the multiplication-formulæ, has not, to my knowledge, hitherto been developed with much completeness or success.

In the second part, Jacobi's partial differential equation is derived in a manner which is most probably the one followed by Jacobi

* Betti, Annali di Matematica, Vol. IV, p. 32.

** Cayley, Cambridge and Dublin Mathematical Journal, Vol. II, pp. 256-266.

† Briot et Bouquet, Theorie des Fonctions Elliptiques, p. 529.

himself, and then applied to the investigation of the multiplication-formulae. The present paper is thus composed of two parts, each of them having reference to the theory of multiplication but otherwise unconnected.

Throughout the paper, I have adopted the notation of *Fund. Nova*, the only exception being that I write θ_1, θ_2, θ_3, θ instead of θ and H, whereby I follow Jacobi in his lectures.*

Part First.

§. 1.

Denote the fundamental elliptic irrationality by s and the corresponding Riemann's surface by T. Let S stand for an algebraic function of z, one-valued on the surface T, and of the order q, having for its zero-points

$$\delta_\mu \{z_\mu \mid s_\mu\}, \quad \mu = 1, 2, 3, \ldots\ldots q,$$

and for its infinity-points

$$\varepsilon_\nu \{\zeta_\nu \mid \sigma_\nu\}, \quad \nu = 1, 2, 3, \ldots\ldots q,$$

then Abel's theorem is expressed by

$$(1.) \qquad \sum_{\mu=1}^{\mu=q} \frac{dz_\mu}{s_\mu} - \sum_{\nu=1}^{\nu=q} \frac{d\zeta_\nu}{\sigma_\nu} = 0.$$

Now S is necessarily of the form

$$\frac{P + Q\,s}{R},$$

where P, Q, R are integral functions of z of the degree q, $q-2$ and q

* Jacobi, Gesammelte Werke, Bd. I, p. 501.

respectively. The numerator $P+Qs$ must be so determined that it vanishes in q points δ_μ and, besides, in another set of q points

$$\varepsilon'_\nu \{\zeta_\nu \mid -\sigma_\nu\}, \quad \nu = 1, 2, 3, \ldots\ldots q.$$

It will be convenient to write $\delta_{\mu+\nu}$ instead of ε'_ν, and then $P+Qs$ vanishes in $2q$ points

$$\delta_\mu \{z_\mu \mid s_\mu\}, \qquad \mu = 1, 2, 3, \ldots q, q+1, \ldots 2q.$$

Equation (1.) then takes the form

$$(2.) \qquad \sum_{\mu=1}^{\mu=2q} \frac{dz_\mu}{s_\mu} = 0.$$

We now take for the fundamental irrationality Riemann's form

$$\sqrt{z.\, 1-z.\, 1-k^2z}\,.$$

and write

$$2u = \int_0^z \frac{dz}{\sqrt{z.\, 1-z.\, 1-k^2z}},$$

so that

$$\sqrt{z} = \operatorname{sn} u\,.$$

Two cases are to be distinguished according as n is odd or even.

§. 2.

When n is odd, put $n=2m+1=2q-1$, and let one of the $2q$ points in which $P+Qs$ vanishes, coincide with the point $\{\zeta \mid \sigma\}$ and the remaining $(2q-1)$ points with the point $\{z \mid s\}$. Equation (2.) then becomes

$$(3)\qquad \frac{d\zeta}{\sigma}+n\frac{dz}{s}=0,\quad n \text{ odd},$$

and, writing $P+Qs$ in full,

$$P+Qs=(a_0+a_1z+a_2z^2+\ldots+a_{m+1}z^{m+1})+(b_0+b_1z+b_2z^2+\ldots+b_{m-1}z^{m-1})s,$$

we must have, denoting differentiation with respect to z by D,

$$(4)\left\{\begin{array}{l} a_0+a_1\zeta+a_2\zeta^2+\ldots+a_{m+1}\zeta^{m+1}+b_0\sigma+b_1(\sigma\zeta)+\ldots+b_{m-1}(\sigma\zeta^{m-1})=0,\\ a_0+a_1z+a_2z^2+\ldots+a_{m+1}z^{m+1}+b_0s+b_1(sz)+\ldots+b_{m-1}(sz^{m-1})=0,\\ \qquad a_1+a_22z+\ldots+a_{m+1}(m+1)z^m+b_0Ds+b_1D(sz)+\ldots+b_{m-1}D(sz^{m-1})=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \qquad\qquad b_0D^{m+2}s+b_1D^{m+2}(sz)+\ldots\\ \qquad\qquad\qquad +b_{m-1}D^{m+2}(sz^{m-1})=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \qquad\qquad b_0D^{2m}s+b_1D^{2m}(sz)+\ldots\\ \qquad\qquad\qquad +b_{m-1}D^{2m}(sz^{m-1})=0.\end{array}\right.$$

Since $a_0, a_1, \ldots\ldots, b_0, b_1 \ldots$ do not all vanish, the determinant obtained by eliminating $a_0, a_1, \ldots\ldots, b_0, b_1 \ldots$ must vanish, that is,

$$(5)\qquad \left|\begin{array}{llllll} 1,\ \zeta,\ \zeta^2,\ \ldots\ldots\ \zeta^{m+1}, & \sigma, & \sigma\zeta, & \sigma\zeta^2, & \ldots\ldots & \sigma\zeta^{m-1}\\ 1,\ z,\ z^2,\ \ldots\ldots\ z^{m+1}, & s, & sz, & sz^2, & \ldots\ldots & sz^{m-1}\\ \quad 1,\ 2z,\ \ldots\ldots(m+1)z^m, & Ds, & D(sz), & D(sz^2), & \ldots\ldots & D(sz^{m-1})\\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ \qquad (m+1)\,!\,, & D^{m+1}s, & D^{m+1}(sz), & D^{m+1}(sz^2), & \ldots\ldots & D^{m+1}(sz^{m-1})\\ & D^{m+2}s, & D^{m+2}(sz), & D^{m+2}(sz^2), & \ldots\ldots & D^{m+2}(sz^{m-1})\\ & D^{m+3}s, & D^{m+3}(sz), & D^{m+3}(sz^2), & \ldots\ldots & D^{m+3}(sz^{m-1})\\ & \ldots\ldots & & & & \\ & D^{2m}s, & D^{2m}(sz), & D^{2m}(sz^2), & \ldots\ldots & D^{2m}(sz^{m-1})\end{array}\right|=0.$$

Let the expansion of this determinant according to the elements of the first row be written,

$$(6)\qquad \{P_0+P_1\zeta+\ldots+P_{m+1}\zeta^{m+1}\}+\{Q_0+Q_1\zeta+\ldots+Q_{m-1}\zeta^{m-1}\}\sigma=0.$$

Then $(2m+2)$ roots of the equation

$$\{P_0+P_1Z+\ldots+P_{m+1}Z^{m+1}\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2.Z(1-Z)(1-k^2Z)=0$$

are ζ, and z repeated $(2m+1)$ times. Thus we obtain the following identity :

$$(7)\qquad \{P_0+P_1Z+\ldots+P_{m+1}Z^{m+1}\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2.Z(1-Z)(1-k^2Z)$$
$$=P_{m+1}^2(Z-\zeta)(Z-z)^{2m+1}.$$

Herein putting $Z=0$, $Z=1$, $Z=\frac{1}{k^2}$ successively and reducing, we obtain

$$(8)\quad \begin{cases} \sqrt{\zeta}\;\; z^{\frac{2m+1}{2}} = \dfrac{P_0}{P_{m+1}}, \\[2ex] \sqrt{1-\zeta}\;\; (1-z)^{\frac{2m+1}{2}} = \dfrac{P_0+P_1+\ldots+P_{m+1}}{P_{m+1}}, \\[2ex] \sqrt{1-k^2\zeta}\;\; (1-k^2z)^{\frac{2m+1}{2}} = \dfrac{P_0k^{2m+2}+P_1k^{2m}+\ldots+P_{m+1}}{P_{m+1}}. \end{cases}$$

In extracting square root, strictly speaking, we have to prefix the double sign $\pm$; but by taking some particular value of n or by putting $k=0$ in the final results to be hereafter obtained, it comes out that we have to take the $+$ sign.

§ 3.

Let us now investigate the expressions which occur on the right-hand side of equation (8). For this purpose, put

$$(9.)\qquad \Delta = \left|\begin{array}{llll} D^{m+1}s, & D^{m+1}sz, & \ldots & D^{m+1}sz^{m-1} \\ D^{m+2}s, & D^{m+2}sz, & \ldots\ldots & D^{m+2}sz^{m-1} \\ \ldots & \ldots & \ldots & \ldots \\ D^{2m}s, & D^{2m}sz, & \ldots\ldots & D^{2m}sz^{m-1} \end{array}\right|;$$

further let the expansion of Δ according to the elements of the first row be written :

$$(10.)\qquad \Delta = D^{m+1}s\,.\,\Delta_1 + D^{m+1}sz\,.\,\Delta_2 + \ldots + D^{m+1}sz^{m-1}\,.\,\Delta_m\,.$$

Now

$$(11.)\qquad P_0 = \left|\begin{array}{llllll} z,\ z^2,\ \ldots & z^{m+1}, & s, & sz, & \ldots\ldots & sz^{m-1} \\ 1,\ 2z,\ldots & (m+1)z^m, & Ds, & Dsz, & \ldots\ldots & Dsz^{m-1} \\ & \ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ & (m+1)\,!\,, & D^{m+1}s, & D^{m+1}sz, & \ldots\ldots & D^{m+1}sz^{m-1} \\ & & D^{m+2}s, & D^{m+2}sz, & \ldots\ldots & D^{m+2}sz^{m-1} \\ & & \ldots & \ldots & \ldots & \ldots \\ & & D^{2m}s, & D^{2m}sz, & \ldots\ldots & D^{2m}sz^{m-1} \end{array}\right|$$

$$= \Delta \left|\begin{array}{ll} z,\ z^2,\ \ldots\ldots\ z^m, & z^{m+1} \\ 1,\ 2z,\ldots\ldots mz^{m-1}, & (m+1)z^m \\ \ldots\ldots\ldots\ldots & \ldots \\ m\,!\,, & (m+1)!\,z \end{array}\right| - (m+1)!\,\Delta_1 \left|\begin{array}{ll} z,\ z^2,\ \ldots\ldots\ z^m, & s \\ 1,\ 2z,\ldots\ldots mz^{m-1}, & Ds \\ \ldots\ldots\ldots\ldots & \ldots \\ m\,!\,, & D^m s \end{array}\right|$$

$$- (m+1)!\,\Delta_2 \left|\begin{array}{ll} z,\ z^2,\ \ldots\ldots\ z^m, & sz \\ 1,\ 2z,\ldots\ldots mz^{m-1}, & Dsz \\ \ldots\ldots\ldots\ldots & \ldots \\ m\,!\,, & D^m sz \end{array}\right|$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (m+1)!\,\Delta_m \left|\begin{array}{ll} z,\ z^2,\ \ldots\ldots\ z^m, & sz^{m-1} \\ 1,\ 2z,\ldots\ldots mz^{m-1}, & Dsz^{m-1} \\ \ldots\ldots\ldots\ldots & \ldots \\ m\,!\,, & D^m sz^{m-1} \end{array}\right|.$$

Writing, for shortness,

$$1!\,2!\ldots m! = m!!,$$

we have

$$\begin{vmatrix} z, & z^2, & \ldots\ldots & z^m, & z^{m+1} \\ 1, & 2z, & \ldots\ldots & mz^{m-1}, & (m+1)z^m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & m!, & (m+1)!\,z \end{vmatrix} = m!!\,z^{m+1};$$

also

$$\begin{vmatrix} z, & z^2, & \ldots & z^m, & s \\ 1, & 2z, & \ldots & mz^{m-1}, & Ds \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & m!, & D^m s \end{vmatrix} = \begin{vmatrix} 1, & 1, & \ldots & 1, & s \\ 1, & 2, & \ldots & m, & zDs \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & m!, & z^m D^m s \end{vmatrix}$$

$$= (m-1)!!\,\{z^m D^m s - m z^{m-1} D^{m-1} s + \ldots + (-1)^m m!\,s\}$$

$$= (m-1)!!\,z^{m+1} D^m\left(\frac{s}{z}\right);$$

similarly

$$\begin{vmatrix} z, & z^2, & \ldots\ldots & z^m, & sz \\ 1, & 2z, & \ldots & mz^{m-1}, & Dsz \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & m!, & D^m sz \end{vmatrix} = (m-1)!!\,z^{m+1} D^m s,$$

$$\begin{vmatrix} z, & z^2, & \ldots\ldots & z^m, & sz^2 \\ 1, & 2z, & \ldots & mz^{m-1}, & Dsz^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & m!, & D^m sz^2 \end{vmatrix} = (m-1)!!\,z^{m+1} D^m sz,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{vmatrix} z, & z^2, \ldots & z^m, & s z^{m-1} \\ 1, & 2z, \ldots & m z^{m-1}, & D(s z^{m-1}) \\ & \ldots\ldots\ldots & & \ldots\ldots\ldots\ldots \\ & & m\,!\,, & D^m(s z^{m-1}) \end{vmatrix} = (m-1)\,!!\; z^{m+1} D^m(s z^{m-2})\,.$$

Hence

$$(12.)\quad P_0 = m\,!!\; z^{m+1}\left\{\Delta - (m+1)\left[D^m\left(\frac{s}{z}\right).\Delta_1 + D^m s.\Delta_2 + \ldots + D^m s z^{m-2}.\Delta_m\right]\right\}.$$

Again

$$(13.)\quad P_{m+1} = (-1)^{m+1}\begin{vmatrix} 1, z, & z^2, \ldots & z^m, & s, & sz, & \ldots\ldots s z^{m-1} \\ & 1, 2z, \ldots & m z^{m-1}, & Ds, & Dsz, & \ldots\ldots D s z^{m-1} \\ & & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ & & m!\,, & D^m s, & D^m s z, & \ldots D^m s z^{m-1} \\ & & & D^{m+1}s, & D^{m+1} s z, & \ldots D^{m+1} s z^{m-1} \\ & & & D^{m+2}s, & D^{m+2} s z, & \ldots D^{m+2} s z^{m-1} \\ & & & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ & & & D^{2m}s, & D^{2m} s z & \ldots D^{2m} s z^{m-1} \end{vmatrix}$$

$$= (-1)^{m+1}\, m\,!!\;\Delta\,.$$

Substituting the values of P_0 and P_{m+1} just found in the first of equations (8.) and dividing by $z^{\frac{2m+1}{2}}$, we get

$$(14.)\quad \sqrt{\zeta} = (-1)^{m+1}\sqrt{z}\,\frac{\Delta - (m+1)\left\{D^m\left(\frac{s}{z}\right).\Delta_1 + D^m s.\Delta_2 + \ldots + D^m s z^{m-2}.\,\Delta_m\right\}}{\Delta}\,.$$

The expression for Δ given in (9.) may be greatly simplified. By an easy reduction, we find

$$(15.)\quad \Delta = \begin{vmatrix} D^{m+1}s, & (m+1)D^m s, & (m+1)D^{m-1}s, & \ldots\ldots\{(m+1)m\ldots 3\}D^2 s \\ D^{m+2}s, & (m+2)D^{m+1}s, & (m+2)(m+1)D^m s, & \ldots\ldots\{(m+2)(m+1)\ldots 4\}D^3 s \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ D^{2m}s, & 2m D^{2m-1}s, & 2m(2m-1)D^{2m-2}s, & \ldots\ldots\{2m(2m-1)\ldots(m+2)\}D^{m+1}s \end{vmatrix},$$

that is,

$$(16.)\quad \Delta = (-1)^{\frac{m(m-1)}{2}} \frac{(2m)\,!!}{m\,!!} \begin{vmatrix} \frac{1}{2\,!}D^2 s, & \frac{1}{3\,!}D^3 s, & \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3\,!}D^3 s, & \frac{1}{4\,!}D^4 s, & \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

The expression

$$(m+1)\left\{D^m \frac{s}{z}.\Delta_1 + D^m s.\Delta_2 + \ \ldots + D^m s z^{m-2}.\Delta_m\right\}$$

which occurs on the righthand side of (12.), admits of being simplified in a similar manner; namely denoting, for a moment, this expression by Δ', we have

$$\Delta' = (m+1) \begin{vmatrix} D^m \frac{s}{z}, & D^m s, & \ldots\ldots & D^m s z^{m-2} \\ D^{m+2}s, & D^{m+2}sz, & \ldots\ldots & D^{m+2}s z^{m-1} \\ \ldots & \ldots & \ldots & \ldots \\ D^{2m}s, & D^{2m}sz, & \ldots\ldots & D^{2m}s z^{m-1} \end{vmatrix}$$

$$= (-1)^{\frac{m(m-1)}{2}} \frac{(2m)\,!!}{m\,!!} \begin{vmatrix} D\frac{s}{z}, & \frac{1}{2\,!}D^2\frac{s}{z}, & \ldots\ldots & \frac{1}{m\,!}D^m\frac{s}{z} \\ \frac{1}{3\,!}D^3 s, & \frac{1}{4\,!}D^4 s, & \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

Leaving the factor $(-1)^{m+1}\sqrt{z}\,.\,(-1)^{\frac{m(m-1)}{2}}\frac{(2m)!!}{m!!}$ out of consideration, the numerator of the righthand side of equation (8.) becomes

$$\begin{vmatrix} \frac{1}{2\,!}D^2s, & \frac{1}{3\,!}D^3s, \dots\dots & \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \dots\dots & \frac{1}{(m+2)!}D^{m+2}s \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \dots\dots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}$$

$$- \begin{vmatrix} D\frac{s}{z}, & \frac{1}{2\,!}D^2\frac{s}{z}, \dots\dots & \frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \dots\dots & \frac{1}{(m+2)!}D^{m+2}s \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \dots\dots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

The two determinants may be compounded together; thus we obtain

$$(17.) \qquad z \begin{vmatrix} \frac{1}{2\,!}D^2\left(\frac{s}{z}\right), & \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), \dots\dots & \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right) \\ \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, \dots\dots & \frac{1}{(m+2)!}D^{m+2}s \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, \dots\dots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

A slightly different form might be given to this expression, viz:

$$(18.) \qquad z^m \begin{vmatrix} \frac{1}{2\,!}D^2\left(\frac{s}{z}\right), & \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), \dots\dots & \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right) \\ \frac{1}{3\,!}D^3\left(\frac{s}{z}\right), & \frac{1}{4\,!}D^4\left(\frac{s}{z}\right), \dots\dots & \frac{1}{(m+2)!}D^{m+2}\left(\frac{s}{z}\right) \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{1}{(m+1)!}D^{m+1}\left(\frac{s}{z}\right), & \frac{1}{(m+2)!}D^{m+2}\left(\frac{s}{z}\right), \dots & \frac{1}{(2m)!}D^{2m}\left(\frac{s}{z}\right) \end{vmatrix}.$$

Let this determinant (omitting the factor z^m) be called M_1 and the determinant on the righthand side of (16.) M. Observe that M and Δ differ from each other only by a numerical factor. Equation (14.) now assumes the form :

$$(19.) \qquad \sqrt{z} = (-1)^{m+1} z^{\frac{n}{2}} \frac{M_1}{M}.$$

§. 4

Consider now

$$P_0 + P_1 + \dots + P_{m+1}$$

which may be written in the form of a determinant, viz.

$$(20.) \qquad \sum_{\mu=0}^{\mu=m+1} P_\mu = \begin{vmatrix} 1, & 1, & 1, & \dots\dots & 1, & 0, & 0, & \dots\dots & 0 \\ 1 & z & z^2 & \dots\dots & z^{m+1}, & s, & sz, & \dots\dots & sz^{m-1} \\ & 1 & 2z & \dots\dots & (m+1)z^m, & Ds, & Dsz, & \dots\dots & Dsz^{m-1} \\ & & & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ & & & & (m+1)!, & D^{m+1}s, & D^{m+1}sz, & \dots\dots & D^{m+1}sz^{m-1} \\ & & & & & D^{m+2}s, & D^{m+2}sz, & \dots\dots & D^{m+2}sz^{m-1} \\ & & & & & \dots\dots & \dots\dots & \dots\dots & \dots \\ & & & & & D^{2m}s, & D^{2m}sz, & \dots\dots & D^{2m}sz^{m-1} \end{vmatrix}.$$

This determinant may be reduced in exactly the same manner as P_0, and then we find

$$\sum_{\mu=0}^{\mu=m+1} P\mu = \Delta \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 1 \\ 1, z, & z^2, \ldots\ldots & z^m, z^{m+1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, (m+1)z^m \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m\,!, (m+1)\,!\,z \end{vmatrix}$$

$$-(m+1)\,!\,\Delta_1 \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, s \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Ds \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m\,!, D^m s \end{vmatrix}$$

$$-(m+1)\,!\,\Delta_2 \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Dsz \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m\,!, D^m sz \end{vmatrix}$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(m+1)!\,\Delta_m \begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz^{m-1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, Dsz^{m-1} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m\,!, D^m sz^{m-1} \end{vmatrix}.$$

Now

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 1 \\ 1, z, & z^2, \ldots\ldots & z^m, z^{m+1} \\ & 1, 2z, \ldots\ldots & mz^{m-1}, (m+1)z^m \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m\,!, (m+1)\,!\,z \end{vmatrix} = (-1)^{m+1} m\,!!\,(1-z)^{m+1},$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, s \\ & 1, 2z, \ldots\ldots mz^{m-1}, & Ds \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m s \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{s}{1-z},$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz \\ & 1, 2z, \ldots\ldots mz^{m-1}, & Dsz \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m sz \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{sz}{1-z},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\begin{vmatrix} 1, 1, & 1, \ldots\ldots & 1, 0 \\ 1, z, & z^2, \ldots\ldots & z^m, sz^{m-1} \\ & 1, 2z, \ldots\ldots mz^{m-1}, & Dsz^{m-1} \\ & \ldots\ldots\ldots\ldots\ldots\ldots & \\ & & m!, D^m sz^{m-1} \end{vmatrix} = -(-1)^{m+1}(m-1)!!(1-z)^{m+1} D^m \frac{sz^{m-1}}{1-z}.$$

Hence

$$\sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{m+1} m!!(1-z)^{m+1} \Big\{ \Delta + (m+1) \Big[D^m \frac{s}{1-z} \cdot \Delta_1 + D^m \frac{sz}{1-z} \cdot \Delta_2 + \ldots$$

$$+ D^m \frac{sz^{m-1}}{1-z} \cdot \Delta_m \Big] \Big\}$$

$$= (-1)^{m+1} m!!(1-z)^{m+2} \begin{vmatrix} D^{m+1} \frac{s}{1-z}, & D^{m+1} \frac{sz}{1-z}, & \ldots\ldots & D^{m+1} \frac{sz^{m-1}}{1-z} \\ D^{m+2} s, & D^{m+2} sz, & \ldots\ldots & D^{m+2} sz^{m-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ D^{2m} s, & D^{2m} sz, & \ldots\ldots & D^{2m} sz^{m-1} \end{vmatrix}.$$

Further, we deduce without much difficulty

$$(21.)\qquad \sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{\frac{m(m-1)}{2}}(-1)^{m+1}(2m)!!(1-z)^{m+2}$$

$$\times\begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3 s, & \frac{1}{4\,!}D^4 s, & \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}s \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \ldots\ldots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix},$$

or

$$(22.)\qquad \sum_{\mu=0}^{\mu=m+1} P_\mu = (-1)^{\frac{m(m-1)}{2}}(-1)^{m+1}(2m)!!(1-z)^{2m+1}$$

$$\times\begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \ldots\ldots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3\frac{s}{1-z}, & \frac{1}{4\,!}D^4\frac{s}{1-z}, & \ldots\ldots & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z}, & \ldots & \frac{1}{(2m)!}D^{2m}\frac{s}{1-z} \end{vmatrix}.$$

Denoting the determinant on the righthand side of (22.) by M_2 and remembering

$$P_{m+1} = (-1)^{m+1}(2m)!!\,M,$$

we get by division

$$\frac{P_0+P_1+\ldots+P_{m+1}}{P_{m+1}} = (1-z)^{2m+1}\frac{M_2}{M}.$$

Substituting this in the second of equations (8.), we obtain

$$(23.)\qquad \sqrt{1-\zeta} = (1-z)^{\frac{n}{2}}\frac{M_2}{M}.$$

§. 5.

Let us consider the determinant

$$(24.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = \begin{vmatrix} k^{2m+2}, & k^{2m}, & k^{2m-2}, & \dots & 1, & 0, & 0, & \dots 0 \\ 1, & z, & z^2, & \dots & z^{m+1}, & s, & sz, & \dots sz^{m-1} \\ & 1, & 2z, & \dots & (m+1)z^m, & Ds, & Dsz, & \dots Dsz^{m-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & (m+1)!\,, & D^{m+1}s, & D^{m+1}sz, & \dots D^{m+1}sz^{m-1} \\ & & & & & D^{m+2}s, & D^{m+2}sz, & \dots D^{m+2}sz^{m-1} \\ & & & & & \dots & \dots & \dots \\ & & & & & D^{2m}s, & D^{2m}sz, & \dots D^{2m}sz^{m-1} \end{vmatrix}.$$

This determinant is of the same form as the one in (20.), and we find likewise

$$\sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = (-1)^{m+1} m\,!!\,(1-k^2z)^{m+1}\Big\{\Delta + (m+1)k^2\Big[D^m\frac{s}{1-k^2z}\cdot\Delta_1$$

$$+ D^m\frac{sz}{1-k^2z}\cdot\Delta_2 + \dots + D^m\frac{sz^{m-1}}{1-k^2z}\cdot\Delta_m\Big]\Big\}$$

$$= (-1)^{m+1}m!!(1-k^2z)^{m+2}\begin{vmatrix} D^{m+1}\dfrac{s}{1-k^2z}, & D^{m+1}\dfrac{sz}{1-k^2z}, & \dots D^{m+1}\dfrac{sz^{m-1}}{1-k^2z} \\ D^{m+2}s, & D^{m+2}sz, & \dots D^{m+2}sz^{m-1} \\ \dots & \dots & \dots \\ D^{2m}s, & D^{2m}sz, & \dots D^{2m}sz^{m-1} \end{vmatrix};$$

whence

$$(25.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = (-1)^{m+1}(2m)!!(1-k^2z)^{m+2}$$

$$\begin{vmatrix} \dfrac{1}{2\,!}D^2\dfrac{s}{1-k^2z}, & \dfrac{1}{3\,!}D^3\dfrac{s}{1-k^2z}, & \dots \dfrac{1}{(m+1)!}D^{m+1}\dfrac{s}{1-k^2z} \\ \dfrac{1}{3\,!}D^3s, & \dfrac{1}{4\,!}D^4s, & \dots \dfrac{1}{(m+2)!}D^{m+2}s \\ \dots & \dots & \dots \\ \dfrac{1}{(m+1)!}D^{m+1}s, & \dfrac{1}{(m+2)!}D^{m+2}s, & \dots \dfrac{1}{(2m)!}D^{2m}s \end{vmatrix},$$

or

$$(26.)\quad \sum_{\mu=0}^{\mu=m+1} P_\mu k^{2m+2-2\mu} = (-1)^{m+1}(-1)^{\frac{m(m-1)}{2}}(2m)!!\,(1-k^2z)^{2m+1}$$

$$\times \begin{vmatrix} \frac{1}{2!}D^2\frac{s}{1-k^2z}, & \frac{1}{3!}D^3\frac{s}{1-k^2z}, & \cdots\cdots & \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z} \\ \frac{1}{3!}D^3\frac{s}{1-k^2z}, & \frac{1}{4!}D^4\frac{s}{1-k^2z}, & \cdots\cdots & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z}, & \cdots & \frac{1}{(2m)!}D^{2m}\frac{s}{1-k^2z} \end{vmatrix}.$$

Call the determinant on the righthand side of (26.) M_3, we have

$$\frac{P_0k^{2m+2}+P_1k^{2m}+\ldots+P_{m+1}}{P_{m+1}} = (1-k^2z)^{2m+1}\frac{M_3}{M}.$$

Substituting this in the last of equations (8.) we obtain

$$(27.)\qquad \sqrt{1-k^2\zeta} = (1-k^2z)^{\frac{n}{2}}\frac{M_3}{M}.$$

§. 6.

In case n is even, say $n=2m$, put $m=q-1$, and, referring to §. 1., let one of the $2q$ points in which $P+Qs$ vanishes, coincide with the point $\{\zeta \mid \sigma\}$, another point with $\{0 \mid 0\}$, and the remaining $(2q-2)$ points with the point $\{z \mid s\}$. Then

$$(28.)\qquad \frac{d\zeta}{\sigma}+n\frac{dz}{s}=0 \quad , \quad n \text{ even,}$$

and

$$P+Qs=(a_1z+a_2z^2+\ldots+a_{m+1}z^{m+1})+(b_0+b_1z+\ldots+b_{m-1}z^{m-1})s,$$

a_0 being zero. If, as will be convenient, we write

$$\frac{\sigma}{\zeta} = \sqrt{\frac{1-\zeta\,.\,1-k^2\zeta}{\zeta}} = \phi\,,\ \frac{s}{z} = \sqrt{\frac{1-z\,.\,1-k^2z}{z}} = u,$$

then $$\frac{P+Qs}{z} = (a_1+a_2z+\ldots+a_{m+1}z^m) + (b_0+b_1z+\ldots+b_{m-1}z)\,u\,,$$

and

$$(29.)\left\{\begin{array}{l}
a_1+a_2\zeta+a_3\zeta^2+\ldots+a_{m+1}\zeta^m+b_0\phi+b_1(\phi\zeta)+\ldots+b_{m-1}(\phi\zeta^{m-1})=0,\\
a_1+a_2z+a_3z^2+\ldots+a_{m+1}z^m+b_0u+b_1(uz)+\ldots+b_{m-1}(uz^{m-1})=0,\\
\quad a_2+a_3.2z+\ldots+a_{m+1}.mz^{m-1}+b_0Du+b_1D(uz)+\ldots+b_{m-1}D(uz^{m-1})=0,\\
\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\qquad\qquad a_{m+1}.\,m!\ +b_0D^m\,u+b_1\,D^m\,(uz)+\ldots\\
\qquad\qquad\qquad\qquad +b_{m-1}D^m\,(uz^{m-1})=0,\\
\qquad\qquad b_0D^{m+1}\,u+b_1D^{m+1}\,(uz)+\ldots\\
\qquad\qquad\qquad\qquad +b_{m-1}D^{m+1}\,(uz^{m-1})=0,\\
\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\qquad\qquad b_0D^{2m-1}u+b_1D^{2m-1}(uz)+\ldots\\
\qquad\qquad\qquad\qquad +b_{m-1}D^{2m-1}(uz^{m-1})=0\,;
\end{array}\right.$$

whence

$$(30.)\quad\left|\begin{array}{lllll}
1,\ \zeta,\ \zeta^2,\ \ldots\ \zeta^m, & \phi, & \phi\zeta, & \ldots & \phi\zeta^{m-1}\\
1,\ z,\ z^2,\ \ldots\ z^m, & u, & uz, & \ldots & uz^{m-1}\\
\quad 1,\ 2z,\ \ldots\ mz^{m-1}, & Du, & Duz, & \ldots & Duz^{m-1}\\
\ldots\ldots\ldots\ldots & \ldots & \ldots\ldots & \ldots & \ldots\ldots\\
\qquad\qquad m!\,, & D^mu, & D^muz, & \ldots & D^muz^{m-1}\\
 & D^{m+1}u, & D^{m+1}uz, & \ldots & D^{m+1}uz^{m-1}\\
 & D^{m+2}u, & D^{m+2}uz, & \ldots & D^{m+2}uz^{m-1}\\
 & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots\\
 & D^{2m-1}u, & D^{2m-1}uz, & \ldots & D^{2m-1}uz^{m-1}
\end{array}\right| = 0\,.$$

The expansion of this determinant according to the elements of the first row may be written :

$$(31.)\quad \{P_1+P_2\zeta+\ldots+P_{m+1}\zeta^m\}+\{Q_0+Q_1\zeta+\ldots+Q_{m-1}\zeta^{m-1}\}\,\phi=0\,.$$

Hence

$$(32.)\quad Z\{P_1+P_2Z+\ldots+P_{m+1}Z^m\}^2-\{Q_0+Q_1Z+\ldots+Q_{m-1}Z^{m-1}\}^2(1-Z)(1-k^2Z)$$
$$=P^2_{m+1}\,(Z-\zeta)(Z-z)^{2m}\,.$$

Herein putting $Z=0$, $Z=1$, $Z=\frac{1}{k^2}$ successively, we obtain, after slight reduction,

$$(33.)\quad \begin{cases} \sqrt{\zeta}\,z^m=\dfrac{Q_0}{P_{m+1}}\,, \\[2ex] \sqrt{1-\zeta}\,(1-z)^m=\dfrac{P_1+P_2+\ldots+P_{m+1}}{P_{m+1}}\,, \\[2ex] \sqrt{1-k^2\zeta}\,(1-k^2z)^m=\dfrac{P_1k^{2m}+P_2k^{2m-2}+\ldots+P_{m+1}}{P_{m+1}}\,. \end{cases}$$

§. 7.

The reduction of the expressions which occur on the righthand side of (33.) runs on the same line as the reduction of the corresponding expressions when n is odd, discussed somewhat in detail in preceding sections. We may therefore at once write down the results:

$$(34.)\quad P_{m+1}=(-1)^{\frac{m(m-1)}{2}}(2m-1)!!\begin{vmatrix} \frac{1}{1!}D\frac{s}{z}, & \frac{1}{2!}D^2\frac{s}{z}, & \ldots\frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{2!}D^2\frac{s}{z}, & \frac{1}{3!}D^3\frac{s}{z}, & \ldots\frac{1}{(m+1)!}D^{m+1}\frac{s}{z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z}, & \ldots\frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z} \end{vmatrix},$$

(35.)

$$Q_0=(-1)^{\frac{(m-1)(m-2)}{2}}(2m-1)!!\begin{vmatrix} \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, & \cdots\frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{4\,!}D^4s, & \frac{1}{5\,!}D^5s, & \cdots\frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots\frac{1}{(2m-1)!}D^{2m-1}s \end{vmatrix},$$

(36.)

$$\sum_{\mu=1}^{\mu=m+1}P_\mu=(-1)^{\frac{m(m-1)}{2}}(2m-1)!!(1-z)^{2m}$$

$$\times\begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \cdots\frac{1}{m!}D^m\frac{s}{z(1-z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-z)}, & \cdots\frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)}, & \cdots\frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-z)} \end{vmatrix},$$

(37.)

$$\sum_{\mu=1}^{\mu=m+1}P_\mu k^{2m+2-2\mu}=(-1)^{\frac{m(m-1)}{2}}(2m-1)!!(1-k^2z)^{2m}$$

$$\times\begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-k^2z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \cdots\frac{1}{m!}D^m\frac{s}{z(1-k^2z)} \\ \frac{1}{2\,!}D^3\frac{s}{z(1-k^2z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-k^2z)}, & \cdots\frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-k^2z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)}, & \cdots\frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-k^2z)} \end{vmatrix}.$$

Calling the determinants on the right-hand sides of equations (34–37.) N, N_1, N_2, N_3 in order, and, substituting in (33.), we obtain

(38.)

$$\begin{cases} \sqrt{\zeta} = (-1)^{m-1}\dfrac{1}{z^m}\dfrac{N_1}{N}, \\ \sqrt{1-\zeta} = (1-z)^m\dfrac{N_2}{N}, \\ \sqrt{1-k^2\zeta} = (1-k^2z)^m\dfrac{N_3}{N}. \end{cases}$$

§. 8.

Writing, as is usual,

$$2u = \int_0^z \frac{dz}{s},$$

so that

$$\sqrt{z} = \operatorname{sn} u,$$

we have, in virtue of equations (3.) and (28.),

$$\begin{aligned} \sqrt{\zeta} &= -\operatorname{sn} nu, \\ \sqrt{1-\zeta} &= \operatorname{cn} nu, \\ \sqrt{1-k^2\zeta} &= \operatorname{dn} nu, \end{aligned}$$

where n is any even or odd integer.

When n is odd, say $n = 2m + 1$, we have from (16.), (19.), (23.) and (27.)

$$(39.) \qquad \left\{ \begin{aligned} \operatorname{sn} nu &= (-1)^m z^{\frac{n}{2}} \frac{M_1}{M}, \\ \operatorname{cn} nu &= (1-z)^{\frac{n}{2}} \frac{M_2}{M}, \\ \operatorname{dn} nu &= (1-k^2 z)^{\frac{n}{2}} \frac{M_3}{M}, \end{aligned} \right.$$

where

$$(40.) \qquad M_1 = \begin{vmatrix} \frac{1}{2\,!} D^2 \frac{s}{z}, & \frac{1}{3\,!} D^3 \frac{s}{z}, & \dots \frac{1}{(m+1)!} D^{m+1} \frac{s}{z} \\ \frac{1}{3\,!} D^3 \frac{s}{z}, & \frac{1}{4\,!} D^4 \frac{s}{z}, & \dots \frac{1}{(m+2)!} D^{m+2} \frac{s}{z} \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{1}{(m+1)!} D^{m+1} \frac{s}{z}, & \frac{1}{(m+2)!} D^{m+2} \frac{s}{z}, & \dots \frac{1}{(2m)!} D^{2m} \frac{s}{z} \end{vmatrix},$$

$$(41.)\quad M_2 = \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-z}, & \frac{1}{3\,!}D^3\frac{s}{1-z}, & \cdots\frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z} \\ \frac{1}{3\,!}D^3\frac{s}{1-z}, & \frac{1}{4\,!}D^4\frac{s}{1-z}, & \cdots\frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-z}, & \cdots\frac{1}{(2m)!}D^{2m}\frac{s}{1-z} \end{vmatrix},$$

$$(42.)\quad M_3 = \begin{vmatrix} \frac{1}{2\,!}D^2\frac{s}{1-k^2z}, & \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \cdots\frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z} \\ \frac{1}{3\,!}D^3\frac{s}{1-k^2z}, & \frac{1}{4\,!}D^4\frac{s}{1-k^2z}, & \cdots\frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}\frac{s}{1-k^2z}, & \frac{1}{(m+2)!}D^{m+2}\frac{s}{1-k^2z}, & \cdots\frac{1}{(2m)!}D^{2m}\frac{s}{1-k^2z} \end{vmatrix},$$

$$(43.)\quad M = \begin{vmatrix} \frac{1}{2\,!}D^2 s, & \frac{1}{3\,!}D^3 s, & \cdots\cdots\frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{3\,!}D^3 s, & \frac{1}{4\,!}D^4 s, & \cdots\cdots\frac{1}{(m+2)!}D^{m+2}s \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots\cdots\frac{1}{(2m)!}D^{2m}s \end{vmatrix}.$$

In case n is even, say $n=2m$, we have, in virtue of (38.),

$$(44.)\quad \begin{cases} \operatorname{sn} nu = (-1)^m\dfrac{1}{z^m}\,\dfrac{N_1}{N}, \\ \operatorname{cn} nu = (1-z)^m\,\dfrac{N_2}{N}, \\ \operatorname{dn} nu = (1-k^2z)^m\,\dfrac{N_3}{N}, \end{cases}$$

where

$$(45.)\qquad N_1 = \begin{vmatrix} \frac{1}{3\,!}D^3s, & \frac{1}{4\,!}D^4s, & \cdots \frac{1}{(m+1)!}D^{m+1}s \\ \frac{1}{4\,!}D^4s, & \frac{1}{5\,!}D^5s, & \cdots \frac{1}{(m+2)!}D^{m+2}s \\ \cdots & \cdots & \cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \frac{1}{(m+2)!}D^{m+2}s, & \cdots \frac{1}{(2m-1)!}D^{2m-1}s \end{vmatrix},$$

$$(46.)\qquad N_2 = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)} \\ \cdots & \cdots & \cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-z)} \end{vmatrix},$$

$$(47.)\qquad N_3 = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z(1-k^2z)}, & \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{m!}D^m\frac{s}{z(1-k^2z)} \\ \frac{1}{2\,!}D^2\frac{s}{z(1-k^2z)}, & \frac{1}{3\,!}D^3\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)} \\ \cdots & \cdots & \cdots \\ \frac{1}{m!}D^m\frac{s}{z(1-k^2z)}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z(1-k^2z)}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z(1-k^2z)} \end{vmatrix},$$

$$(48.)\qquad N = \begin{vmatrix} \frac{1}{1\,!}D\frac{s}{z}, & \frac{1}{2\,!}D^2\frac{s}{z}, & \cdots \frac{1}{m!}D^m\frac{s}{z} \\ \frac{1}{2\,!}D^2\frac{s}{z}, & \frac{1}{3\,!}D^3\frac{s}{z}, & \cdots \frac{1}{(m+1)!}D^{m+1}\frac{s}{z} \\ \cdots & \cdots & \cdots \\ \frac{1}{m!}D^m\frac{s}{z}, & \frac{1}{(m+1)!}D^{m+1}\frac{s}{z}, & \cdots \frac{1}{(2m-1)!}D^{2m-1}\frac{s}{z} \end{vmatrix}.$$

It will be observed that the formulæ of $\operatorname{sn} nu$ for $n=2,\ 3,\ 4,\ 5,$ are those given by Jacobi in the Memoir referred to, which appeared

on the opening page of the present paper. I was not aware of the existence of Jacobi's formulæ when I first found the formula for $\mathrm{sn}\, mu$ which was, in fact, then obtained in a slightly different form as furnished by (17.), and which is perhaps in some respects preferable to the one here given. I have, however, given it its present shape, in order that, for the particular values of n, it may exactly coincide with the formulæ given by the illustrious mathematician whose memory is sacred to every student of the theory of elliptic functions.

§. 9.

Before proceeding further with the reduction of the multiplication-formulæ, it is necessary to give a few formulæ relating to the differentiation of composite functions, to which frequent reference will subsequently be made.

Let u be a function of y and y a function of x. It is required to find the n^{th} differential coefficient of u with respect to x. By actual differentiation, we find

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx},$$

$$\frac{d^2u}{dx^2} = \frac{du}{dy}\frac{d^2y}{dx^2} + \frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2,$$

$$\frac{d^3u}{dx^3} = \frac{du}{dy}\frac{d^3y}{dx^3} + 3\frac{d^2u}{dy^2}\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^3,$$

$$\frac{d^4u}{dx^4} = \frac{du}{dy}\frac{d^4y}{dx^4} + \frac{d^2u}{dy^2}\left\{4\frac{dy}{dx}\frac{d^3y}{dx^3} + 3\left(\frac{d\,y}{dx^2}\right)^2\right\} + 6\frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}$$
$$+ \frac{d^4u}{dy^4}\left(\frac{dy}{dx}\right)^4,$$

$$\frac{d^5u}{dx^5} = \frac{du}{dy}\frac{d^5y}{dx^5} + \frac{d^2u}{dy^2}\left\{5\frac{dy}{dx}\frac{d^4y}{dx^4} + 10\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\right\}$$

$$+ \frac{d^3u}{dy^3}\left\{10\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3} + 15\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2\right\}$$

$$+ 10\frac{d^4u}{dy^4}\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2} + \frac{d^5u}{dy^5}\left(\frac{dy}{dx}\right)^5,$$

....;

and, generally,

(49) $$\frac{d^nu}{dx^n} = \frac{du}{dy}X_1 + \frac{d^2u}{dy^2}X_2 + \ldots + \frac{d^ru}{dy^r}X_r + \ldots + \frac{d^{n-1}u}{dy^{n-1}}X_{n-1} + \frac{d^nu}{dy^n}X_n,$$

where $X_1, X_2, \ldots X_n$ denote as yet unknown functions which contain only the differential coefficients of y with respect to x. A few of the initial and end coefficients may at once be written down:

(50) $$\left\{\begin{aligned}
X_1 &= \frac{d^ny}{dx^n},\\
X_2 &= n\frac{dy}{dx}\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2y}{dx^2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\\
&\quad + \begin{cases} \dfrac{n(n-1)\ldots(n-\frac{n-1}{2}+1)}{\left(\frac{n-1}{2}\right)!}\dfrac{d^{\frac{n-1}{2}}y}{dx^{\frac{n-1}{2}}}\cdot\dfrac{d^{\frac{n+1}{2}}y}{dx^{\frac{n+1}{2}}}, & (n \text{ odd})\\[2ex] \dfrac{n(n-1)\ldots(n-\frac{n}{2}+1)}{2\cdot\left(\frac{n}{2}\right)!}\left(\dfrac{d^{\frac{n}{2}}y}{dx^{\frac{n}{2}}}\right)^2, & (n \text{ even})\end{cases}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
X_{n-2} &= \frac{n(n-1)(n-2)}{3!}\left(\frac{dy}{dx}\right)^{n-3}\frac{d^3y}{dx^3}\\
&\quad + \frac{n(n-1)(n-2)(n-3)}{2^3}\left(\frac{dy}{dx}\right)^{n-4}\left(\frac{d^2y}{dx^2}\right)^2,\\
X_{n-1} &= \frac{n(n-1)}{2!}\left(\frac{dy}{dx}\right)^{n-2}\frac{d^2y}{dx^2},\\
X_n &= \left(\frac{dy}{dx}\right)^n.
\end{aligned}\right.$$

To find X_r, observe that X_r, being a function of the differential coefficients of y with respect to x only, is independent of u as a function of y. Hence, putting $u=y, y^2, \ldots y^n$ successively, we obtain

$$\begin{cases} \dfrac{d^n y}{dx^n} = X_1 , \\ \dfrac{d^n(y^2)}{dx^n} = 2yX_1+2X_2 , \\ \dfrac{d^n(y^3)}{dx^n} = 3y^2X_1+3.2yX_2+3.2X_3 , \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d^n(y^r)}{dx^n} = ry^{r-1}X_1+r(r-1)y^{r-2}X_2+\ldots+r!\,X_r , \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d^n(y^n)}{dx^n} = ny^{n-1}X_1+n(n-1)y^{n-2}X_2+ \dots\dots\dots +n!\,X_n . \end{cases}$$

To obtain $X_1, X_2, \ldots$, we may solve these equations as was done by Bertrand* ; but it is shorter to proceed as follows:—Put $u=e^{\lambda y}$, then

$$e^{-\lambda y}\frac{d^n e^{\lambda y}}{dx^n} = \lambda X_1+\lambda^2X_2+\ldots+\lambda^r X_r+\ldots+\lambda^n X_n ,$$

and, on the other hand,

$$e^{-\lambda y} = 1-\lambda y+\frac{\lambda^2y^2}{2\,!}-\frac{\lambda^3y^3}{3\,!}+\ldots+(-1)^n\frac{\lambda^ny^n}{n\,!}+\ldots ,$$

$$\frac{d^n e^{\lambda y}}{dx^n} = \lambda\frac{d^ny}{dx^n}+\frac{\lambda^2}{2\,!}\frac{d^ny^2}{dx^n}+\frac{\lambda^3}{3\,!}\frac{d^ny^3}{dx^n}+\ldots+\frac{\lambda^n}{n\,!}\frac{d^ny^n}{dx^n}+\ldots .$$

Multiplying the last two equations together and equating the coefficients of the like powers of λ, we obtain

* Bertrand, Traité de Calcul Différential et de Calcul Integral, T. I, p. 139.

$$
(51)\quad \left\{
\begin{aligned}
X_1 &= \frac{d^n y}{dx^n}\,,\\
X_2 &= \frac{1}{2\,!}\,\frac{d^n(y^2)}{dx^n} - y\,\frac{d^n y}{dx^n}\,,\\
&\cdots\cdots\cdots\cdots\\
X_r &= \frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n} - \frac{y}{(r-1)!\,1!}\,\frac{d^n(y^{r-1})}{dx^n} + \frac{y^2}{(r-2)!\,2!}\,\frac{d^n(y^{r-2})}{dx^n} - \dots\\
&\qquad\qquad + (-1)^{r-1}\frac{y^{r-1}}{1!\,(r-1)!}\,\frac{d^n y}{dx^n}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
X_n &= \frac{1}{n\,!}\,\frac{d^n(y^n)}{dx^n} - \frac{y}{(n-1)!\,1!}\,\frac{d^n(y^{n-1})}{dx^{n-1}} + \frac{y^2}{(n-2)!\,2!}\,\frac{d^n y^{n-2}}{dx^n} - \dots\\
&\qquad\qquad + (-1)^{n-1}\frac{y^{n-1}}{1!\,(n-1)!}\,\frac{d^n y}{dx^n}\,.
\end{aligned}
\right.
$$

It may be worth while to notice that the expression on the opposite side of X_r is identically zero for all values of r greater than n.

From the mode of derivation it is evident that X_r contains differential coefficients of y but not y itself. Hence, if $\dfrac{1}{r\,!}\,\dfrac{d^n(y^r)}{dx^n}$ be broken up into two parts such that one part contains all the terms independent of explicit y and the other part, terms having y as a factor, then

$$X_r = \text{that part of } \frac{1}{r\,!}\,\frac{d^n(y^r)}{dx^n} \text{ which is independent of explicit } y$$

and that part of $\dfrac{1}{r\,!}\,\dfrac{d^n(y^r)}{dx^n}$ which contains explicit y together with

$$-\frac{y}{(r-1)!\,1!}\,\frac{d^n(y^{r-1})}{dx^n} + \frac{y^2}{(r-2)!\,2!}\,\frac{d^n(y^{r-2})}{dx^n} - \dots + (-1)^n\frac{y^{r-1}}{1!\,(r-1)!}\,\frac{d^n y}{dx^n}$$

is identically zero. Now

$$\frac{1}{n\,!}\,\frac{d^n(y^{r-1})}{dx^n} = \text{coefficient of } h^n \text{ in } \left\{y + h\frac{dy}{dx} + \frac{h^2}{2\,!}\,\frac{d^2y}{dx^2} + \dots\right\}^r.$$

Hence that part of $\frac{1}{r!}\frac{d^n(y^r)}{dx^n}$ which is independent of explicit y is equal to the coefficient of h^n in

$$\frac{n!}{r!}\left\{h\frac{dy}{dx}+\frac{h^2}{2!}\frac{d^2y}{dx^2}+\dots\right\}^r,$$

that is, in

$$\frac{n!}{r!}\left\{f(x+h)-f(x)\right\}^r,$$

where $y=f(x)$.

Thus we obtain

$$(52) \qquad X_r = \frac{1}{r!}\left[\left(\frac{d}{dh}\right)^n\left\{f(x+h)-f(x)\right\}\right]_{h=0}.$$

This form of X_r has been obtained in a different manner by U. Meyer.* Bertrand gives the following form of X_r which is substantially the same as (51) :**

$$X_r = \frac{y^r}{r!}\frac{d^n\left(\frac{y}{\alpha}-1\right)^r}{dx^n},$$

where α is to be regarded as constant during differentiation and is afterwards to be replaced by y. Again making use of the form of X_r given by (51), we get

$$\frac{d^nu}{dx^n}=\sum_{r=1}^{r=n}\frac{1}{r!}\frac{d^ru}{dy^r}\frac{d^ny^r}{dx^n}-y\sum_{r=2}^{r=n}\frac{1}{(r-1)!}\frac{d^ru}{dy^r}\frac{d^ny^{r-1}}{dx^n}$$
$$+y^2\sum_{r=3}^{r=n}\frac{1}{(r-2)!}\frac{d^ru}{dy^r}\frac{d^ny^{r-2}}{dx^n}-\dots+(-1)^{n-1}y^{n-1}\frac{d^nu}{dy^n}\frac{d^ny}{dx^n},$$

* Grunert's Archiv der Mathematik, Bd. IX.
** Loc. cit. p. 140.

which agrees with the form of $\frac{d^n u}{dx^n}$ given by R. Hoppe.*

Put $u = y^p$, then $\frac{d^r u}{dy^r} = p_r y^{p-r}$, where p_r denotes the continued product of r quantities $p, p-1, p-2, \ldots (p-r+1)$. Multiplying equations (51) by py^{p-1}, $p(p-1)y^{p-2}, \ldots$ in order and adding, we obtain

$$(53)\quad \frac{d^n y^p}{dx^n} = \frac{p(n-p)_n}{n!\, y^{n-p}} \left\{ -\frac{n_1}{p-1} y^{n-1} \frac{d^n y}{dx^n} + \frac{n_2}{p-2} \frac{y^{n-2}}{2!} \frac{d^n y^2}{dx^n} - \ldots \right.$$
$$\left. + (-1)^r \frac{n_r}{p-r} \frac{y^{n-r}}{r!} \frac{d^n y^r}{dx^n} + \ldots + (-1)^n \frac{1}{p-n} \frac{d^n y^n}{dx^n} \right\}.$$

For $p = \frac{1}{2}$, we have

$$(54)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{y^{n-\frac{1}{2}}} \left\{ \frac{n_1}{1} y^{n-1} \frac{d^n y}{dx^n} - \frac{n_2}{2!} \frac{y^{n-2}}{3} \frac{d^n y^2}{dx^n} + \ldots \right.$$
$$\left. + (-1)^{r-1} \frac{n_r}{r!} \frac{y^{n-r}}{2r-1} \frac{d^n y^r}{dx^n} + \ldots + (-1)^{n-1} \frac{1}{2n-1} \frac{d^n y^n}{dx^n} \right\}.$$

If y be a rational integral function of x of the third degree, $\frac{d^n y^r}{dx^n}$ vanishes for all values of r for which $3r < n$. In this case, denoting by l either $\frac{n}{3}$ or the integer next above $\frac{n}{3}$ according as n is a multiple of 3 or not, we have

$$(55)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{y^{n-\frac{1}{2}}} \left\{ (-1)^{l-1} \frac{n_l}{l!} \frac{y^{n-l}}{2l-1} \frac{d^n y^l}{dx^n} + \ldots \right.$$
$$\left. + (-1)^{r-1} \frac{n_r}{r!} \frac{y^{n-r}}{2r-1} \frac{d^n y^r}{dx^n} + \ldots + (-1)^{n-1} \frac{1}{2n-1} \frac{d^n y^n}{dx^n} \right\}.$$

Again

$$(56)\quad \frac{d^n \sqrt{y}}{dx^n} = \frac{1}{2^n y^{n-\frac{1}{2}}} \left\{ (2y)^{n-1} X_1 - (2y)^{n-2} X_2 + \ldots \right.$$
$$\left. + (-1)^{r-1}.\ 1.3.\ \ldots (2r-3)(2y)^{n-r} X_r + \ldots + (-1)^n.\ 1.3.\ \ldots (2n-3) X_n \right\},$$

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and if y be a rational integral function of the third degree, $\dfrac{d^4y}{dx^4}$ and all the higher differential coefficients vanish, so that

$$X_r = \text{coef. of } h^n \text{ in } \frac{n\,!}{r\,!}\left\{h\frac{dy}{dx} + \frac{1}{2}h^2\frac{d^2y}{dx^2} + \frac{1}{6}h^3\frac{d^3y}{dx^3}\right\}^r,$$

or

$$X_{n-r} = \text{coef. of } h^r \text{ in } \frac{n\,!}{(n-r)!}\left\{\frac{dy}{dx} + \frac{1}{2}h\frac{d^2y}{dx^2} + \frac{1}{6}h^2\frac{d^3y}{dx^3}\right\}^{n-r};$$

whence follows that X_{n-r} vanishes for all values of r for which $r > 2n-2r$, that is, $3r > 2n$. Denoting the integral part of $\dfrac{2n}{3}$ by i, we have

$$(57)\quad \frac{d^n\sqrt{y}}{dx^n} = (-1)^{n-1}\frac{1}{2^n y^{n-\frac{1}{2}}}\Big\{(-1)^i.1.3.\ldots(2n-2i-3)(2y)^iX_{n-i} + \ldots$$

$$+(-1)^r.1.3.\ldots(2n-2r-3)(2y)^rX_{n-r} + \ldots + 1.3.\ldots(2n-3)X_n\Big\},$$

where

$$X_{n-r} = n\,!\left\{\frac{1}{(n-2r)!\,r!}\left(\frac{dy}{dx}\right)^{n-2r}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^r\right.$$

$$+\frac{1}{(n-2r+1)!\,(r-2)!\,1!}\left(\frac{dy}{dx}\right)^{n-2r+1}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^{r-2}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)$$

$$\left.+\frac{1}{(n-2r+2)!\,(r-4)!\,2!}\left(\frac{dy}{dx}\right)^{n-2r+2}\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)^{r-4}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)^2 + \ldots\right\},$$

the last term being

$$\frac{1}{(n-\frac{3}{2}r)!\,(\frac{r}{2})!}\left(\frac{dy}{dx}\right)^{n-\frac{3r}{2}}\left(\frac{1}{6}\frac{d^3y}{dx^3}\right)^{\frac{r}{2}}$$

or

$$\frac{1}{\left(n-\frac{3r+1}{2}\right)!\,1\,!\left(\frac{r-1}{2}\right)!}\left(\frac{dy}{dx}\right)^{n-\frac{3r+1}{2}}\left(\frac{1}{2}\,\frac{d^2y}{dx^2}\right)\left(\frac{1}{6}\,\frac{d^3y}{dx^3}\right)^{\frac{r-1}{2}},$$

according as r is even or odd.

Similarly

$$(58)\quad \frac{d^ny^{-\frac{1}{2}}}{dx^n} = \frac{(-1)^n}{2^ny^{n+\frac{1}{2}}}\Big\{(-1)^i.1.3.\ldots(2n-2i-1)(2y)^iX_{n-i}+\ldots$$
$$+(-1)^r.1.3.\ldots(2n-2r-1)(2y)^rX_{n-r}+\ldots+1.3.\ldots(2n-1)X_n\Big\},$$

where X_{n-r} has the same signification as in (57).

$$\text{Let}\quad f = s^2 = z(1-z)(1-k^2z) = z-(1+k^2)z^2+k^2z^3\,,$$

$$f_1 = \frac{df}{dz} = 1-2(1+k^2)z+3k^2z^2\,,$$

$$f_2 = \frac{1}{2}\,\frac{d^2f}{dz^2} = -(1+k^2)+3k^2z\,,$$

$$f_3 = \frac{1}{6}\,\frac{d^3f}{dz^3} = k^2\,.$$

Now write

$$(59)\qquad \frac{1}{n\,!}D^n s = \frac{(-1)^{n-1}}{2^n\,s^{2n-1}}S_n\,,$$

$$(60)\qquad \frac{1}{n\,!}D^n\frac{1}{s} = \frac{(-1)^n}{2^n\,s^{2n+1}}R_n\,;$$

then by (57) and (58)

$$(61)\quad S_n = (-1)^i.1.3.\ldots(2n-2i-3)(2f)^iZ_{n-i}+\ldots$$
$$+(-1)^r.1.3.\ldots(2n-2r-3)(2f)^rZ_{n-r}+\ldots+1.3.\ldots(2n-3)Z_n\,,$$

$$(62) \quad R_n = (-1)^i.1.3.\ldots(2n-2i-1)(2f)^i Z_{n-i} + \ldots$$

$$+(-1)^r.1.3.\ldots(2n-2r-1)(2f)^r Z_{n-r} + \ldots + 1.3.\ldots(2n-1)Z_n\,,$$

where

$$(63) \quad Z_{n-r} = \frac{f_1^{n-2r} f_2^r}{(n-2r)!\,r!} + \frac{f_1^{n-2r-1} f_2^{r-2} f_3}{(n-2r+1)!(r-2)!\,1!} + \frac{f_1^{n-2r-2} f_2^{r-4} f_3^2}{(n-2r+2)!\,(r-4)!\,2!} + \cdots,$$

the last term being

$$\frac{f_1^{n-\frac{3r}{2}} f_3^{\frac{r}{2}}}{\left(n-\frac{3}{2}r\right)!\left(\frac{r}{2}\right)!} \quad \text{or} \quad \frac{f_1^{n-\frac{3r+1}{2}} f_2 f_3^{\frac{r-1}{2}}}{\left(n-\frac{3r+1}{2}\right)!\left(\frac{r-1}{2}\right)!}$$

according as r is even or odd.

Observe that both S_n and R_n are rational integral functions of z of the degree $2n$.

§ 10.

The determinant-expressions of § 8, simple as they may appear, are not convenient for introducing S_n and R_n of the last section. It is desirable to derive the multiplication-formulae in forms slightly different from those given in § 8.

Consider first the case n odd. Referring to § 2, we might have taken $\frac{P}{s}+Q$ instead of $P+Qs$ and then we shall have in place of (5) the following determinant :

$$(64)\qquad \begin{vmatrix} 1, \zeta, \zeta^2, \dots \zeta^{m-1}, & \frac{1}{\sigma}, & \frac{\zeta}{\sigma}, & \frac{\zeta^2}{\sigma}, & \dots \frac{\zeta^{m+1}}{\sigma} \\ 1, z, z^2, \dots z^{m-1}, & \frac{1}{s}, & \frac{z}{s}, & \frac{z^2}{s}, & \dots \frac{z^{m+1}}{s} \\ 1, 2z, \dots (m-1)z^{m-2}, & D\frac{1}{s}, & D\frac{z}{s}, & D\frac{z^2}{s}, & \dots D\frac{z^{m+1}}{s} \\ \dots & \dots & \dots & \dots & \dots \\ (m-1)!\,, & D^{m-1}\frac{1}{s}, & D^{m-1}\frac{z}{s}, & D^{m-1}\frac{z^2}{s}, & \dots D^{m-1}\frac{z^{m+1}}{s} \\ & D^{m}\frac{1}{s}, & D^{m}\frac{z}{s}, & D^{m}\frac{z^2}{s}, & \dots D^{m}\frac{z^{m+1}}{s} \\ & D^{m+1}\frac{1}{s}, & D^{m+1}\frac{z}{s}, & D^{m+1}\frac{z^2}{s}, & \dots D^{m+1}\frac{z^{m+1}}{s} \\ & \dots & \dots & \dots & \dots \\ & D^{2m}\frac{1}{s}, & D^{2m}\frac{z}{s}, & D^{2m}\frac{z^2}{s}, & \dots D^{2m}\frac{z^{m+1}}{s} \end{vmatrix} = 0\,.$$

Now put

$$(65)\qquad \Omega = \begin{vmatrix} D^{m-1}\frac{1}{s}, & D^{m-1}\frac{z}{s}, & \dots D^{m-1}\frac{z^{m+1}}{s} \\ D^{m}\frac{1}{s}, & D^{m}\frac{z}{s}, & \dots D^{m}\frac{z^{m+1}}{s} \\ \dots & \dots & \dots \\ D^{2m}\frac{1}{s}, & D^{2m}\frac{z}{s}, & \dots D^{2m}\frac{z^{m+1}}{s} \end{vmatrix},$$

and let the expansion of this determinant according to the elements of the first row be written :

$$\Omega = D^{m-1}\frac{1}{s}\,.\ \Omega_1 + D^{m-1}\frac{z}{s}\,.\ \Omega_2 + \dots + D^{m-1}\frac{z^{m+1}}{s}\,.\ \Omega_{m+2}\,.$$

We find

$$\sqrt{\zeta}\ z^{\frac{2m+1}{2}} = \frac{\Omega_1}{\Omega_{m+2}}\,,$$

$$\sqrt{1-\zeta}\ (1-z)^{\frac{2m+1}{2}} = \frac{\Omega_1+\Omega_2+\dots+\Omega_{m+2}}{\Omega_{m+2}}\,,$$

$$\sqrt{1-k^2\zeta}\ (1-k^2z)^{\frac{2m+1}{2}} = \frac{\Omega_1 k^{2m+2}+\Omega_2 k^{2m}+\dots+\Omega_{m+2}}{\Omega_{m+2}}\,.$$

and thence, n being odd,

$$(66)\quad \begin{cases} \operatorname{sn} nu = (-1)^m \; z^{-\frac{n}{2}} \dfrac{M_1'}{M'}, \\ \operatorname{cn} nu = \qquad (1-z)^{-\frac{n}{2}} \dfrac{M_2'}{M'}, \\ \operatorname{dn} nu = \qquad (1-k^2 z)^{-\frac{n}{2}} \dfrac{M_3'}{M'}, \end{cases}$$

where

$$(67)\qquad M' = \begin{vmatrix} \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dots \dfrac{1}{m!} D^m \dfrac{1}{s} \\ \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{1}{s}, & \dots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s} \\ \dots\dots & \dots\dots & \dots\dots \\ \dfrac{1}{m!} D^m \dfrac{1}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s}, & \dots \dfrac{1}{(2m)!} D^{2m} \dfrac{1}{s} \end{vmatrix},$$

$$(68)\qquad M_1' = \begin{vmatrix} \dfrac{z}{s}, & \dfrac{1}{1\,!} D \dfrac{z}{s}, & \dots \dfrac{1}{m!} D^m \dfrac{z}{s} \\ \dfrac{1}{1\,!} D \dfrac{z}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{z}{s}, & \dots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{z}{s} \\ \dots\dots & \dots\dots & \dots\dots \\ \dfrac{1}{m!} D^m \dfrac{z}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{z}{s}, & \dots \dfrac{1}{(2m)!} D^{2m} \dfrac{z}{s} \end{vmatrix},$$

$$(69)\qquad M_2' = \begin{vmatrix} (1-z)^{m+1}, & (1-z)^m, & (1-z)^{m-1}, & \dots\; 1 \\ 0, & \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dots \dfrac{1}{m!} D^m \dfrac{1}{s} \\ \dfrac{1}{s}, & \dfrac{1}{1\,!} D \dfrac{1}{s}, & \dfrac{1}{2\,!} D^2 \dfrac{1}{s}, & \dots \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dfrac{1}{(m-1)!} D^{m-1} \dfrac{1}{s}, & \dfrac{1}{m!} D^m \dfrac{1}{s}, & \dfrac{1}{(m+1)!} D^{m+1} \dfrac{1}{s}, & \dots \dfrac{1}{(2m)!} D^{2m} \dfrac{1}{s} \end{vmatrix},$$

$$(70)\quad M_3' = \left| \begin{array}{lllll} (1-k^2z)^{m+1}, & k^2(1-k^2z)^m, & k^4(1-k^2z)^{m-1}, & \dots & k^{2m+2} \\ 0, & \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \dots & \frac{1}{m!}D^m\frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \frac{1}{2\,!}D^2\frac{1}{s}, & \dots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \dots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{array} \right|.$$

Observe that the expression of sn mu as given by (67) and (68) is that which has been indicated by Jacobi. It will, however, be more convenient to write M_1' as follows:

$$(71)\quad M_1' = \left| \begin{array}{lllll} z^{m+1}, & -z^m, & z^{m-1}, & \dots & (-1)^{m+1} \\ 0, & \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \dots & \frac{1}{m!}D^m\frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \frac{1}{2\,!}D^2\frac{1}{s}, & \dots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \dots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{array} \right|.$$

We may here point out that the comparison of the preceding formulæ with the corresponding expressions of § 8, gives some elegant theorems in determinants. Take for example M and M'. We find easily

$$M = C\, s^{2m+1} M',$$

where C denotes a constant which may be a function of m. To determine C we may take some particular values of m. For $m = 1$, we see at once that

$$\frac{1}{2\,!}D^2 s = \quad -s^3 \left| \begin{array}{ll} \frac{1}{s}, & D\frac{1}{s}, \\ D\frac{1}{s}, & \frac{1}{2\,!}D^2\frac{1}{s} \end{array} \right|.$$

For $m=2$, we have

$$\begin{vmatrix} \frac{1}{s}, & D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{4!}D^4\frac{1}{s} \end{vmatrix} = s^2 \begin{vmatrix} \frac{1}{s}, & D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{s}D\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{4!}D^4\frac{1}{s} \end{vmatrix}$$

$$= s\begin{vmatrix} \frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}-\left(D\frac{1}{s}\right)^2, & \frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}-D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{s}\cdot\frac{1}{2!}D^3\frac{1}{s}-D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{s}\cdot\frac{1}{4!}D^4\frac{1}{s}-\left(\frac{1}{2}D^2\frac{1}{s}\right)^2 \end{vmatrix}.$$

Multiply the first row by $D\frac{1}{s}$ and subtract the product from the second row, and then, multiply the first column by $D\frac{1}{s}$ and subtract the product from the second column ; and, observing

$$\frac{1}{2!}D^2s = \quad s^3\left(D\frac{1}{s}\right)^2-s^2\cdot\frac{1}{2!}D^2\frac{1}{s},$$

$$\frac{1}{3!}D^3s = -s^4\left(D\frac{1}{s}\right)^3+2s^3D\frac{1}{s}\cdot\frac{1}{2!}D^2\frac{1}{s}-s^2\frac{1}{3!}D^3\frac{1}{s},$$

$$\frac{1}{4!}D^4s = \quad s^5\left(D\frac{1}{s}\right)^4-3s^4\left(D\frac{1}{s}\right)^2\cdot\frac{1}{2!}D^2\frac{1}{s},$$

$$+s^3\left\{2D\frac{1}{s}\cdot\frac{1}{3!}D^3\frac{1}{s}\left(\frac{1}{2!}D^2\frac{1}{s}\right)^2\right\}-s^2\frac{1}{4!}D^4\frac{1}{s},$$

we get

$$\begin{vmatrix} \frac{1}{2!}D^2s, & \frac{1}{3!}D^3s \\ \frac{1}{3!}D^3s, & \frac{1}{4!}D^4s \end{vmatrix} = s^5\begin{vmatrix} \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s} \\ \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s} \\ \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \frac{1}{4!}D^4\frac{1}{s} \end{vmatrix}.$$

Thus we find $C=(-1)^m$ and thence

$$
(72)\qquad \begin{vmatrix} \frac{1}{2\,!}D^2s, & \cdots & \frac{1}{(m+1)!}D^{m+1}s \\ \cdots\cdots & \cdots & \cdots\cdots \\ \frac{1}{(m+1)!}D^{m+1}s, & \cdots & \frac{1}{(2m)!}D^{2m}s \end{vmatrix}
$$

$$
=(-1)^m s^{2m+1}\begin{vmatrix} \frac{1}{s}, & \frac{1}{1\,!}D\frac{1}{s}, & \cdots & \frac{1}{m\,!}D^m\frac{1}{s} \\ \frac{1}{1\,!}D\frac{1}{s}, & \frac{1}{2\,!}D^2\frac{1}{s}, & \cdots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \cdots & \frac{1}{(2m)!}D^{2m}\frac{1}{s} \end{vmatrix},
$$

which identity may also be proved directly in the manner exemplified by the particular case $m=2$.

§ 11.

Consider next the case n even. Referring to § 6, if we take $\frac{P}{s}+Q$ instead of $P+Qs$, we shall have in place of (30) the following determinant :

$$
(73)\qquad
\left|\begin{array}{rrrr}
1,\ \zeta,\ \zeta^2,\ \ldots \qquad \zeta^{m-1}, & \dfrac{\zeta}{\sigma}, & \dfrac{\zeta^2}{\sigma}, & \ldots\ldots \quad \dfrac{\zeta^{m+1}}{\sigma} \\
1,\ z,\ z^2,\ \ldots \qquad z^{m-1}, & \dfrac{z}{s}, & \dfrac{z^2}{s}, & \ldots\ldots \quad \dfrac{z^{m+1}}{s} \\
1,\ 2z, \ldots\ (m-1)\, z^{m-2}, & D\dfrac{z}{s}, & D\dfrac{z^2}{s}, & \ldots\ldots\ D\,\dfrac{z^{m+1}}{s} \\
\ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\
(m-1)\,!\,, & D^{m-1}\dfrac{z}{s}, & D^{m-1}\dfrac{z^2}{s}, & \ldots\ldots D^{m-1}\,\dfrac{z^{m+1}}{s} \\
 & D^{m}\,\dfrac{z}{s}, & D^{m}\,\dfrac{z^2}{s}, & \ldots\ldots\ D^{m}\,\dfrac{z^{m+1}}{s} \\
 & D^{m+1}\dfrac{z}{s}, & D^{m+1}\dfrac{z^2}{s}, & \ldots\ldots D^{m+1}\,\dfrac{z^{m+1}}{s} \\
 & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\
 & D^{2m-1}\dfrac{z}{s}, & D^{2m-1}\dfrac{z^2}{s}, & \ldots\quad D^{2m-1}\dfrac{z^{m+1}}{s}
\end{array}\right| = 0,
$$

say $$Q_0 + Q_1\zeta + Q_2\zeta^2 + \ldots + Q_{m-1}\zeta^{m-1} + P_1\frac{\zeta}{\sigma} + P_2\frac{\zeta^2}{\sigma} + \ldots + P_{m+1}\frac{\zeta^{m+1}}{\sigma} = 0.$$

Now

$$\sqrt{\zeta}\, z^m = \frac{Q_0}{P_{m+1}},$$

$$\sqrt{1-\zeta}\,(1-z)^m = \frac{P_1 + P_2 + \ldots + P_{m+1}}{P_{m+1}},$$

$$\sqrt{1-k^2\zeta}\,(1-k^2 z)^m = \frac{P_1 k^{2m} + P_2 k^{2m-2} + \ldots + P_{m+1}}{P_{m+1}},$$

and thence, n being even,

$$
(74)\qquad
\begin{cases}
\operatorname{sn} nu = (-1)^m\, z^{\frac{n}{2}}\ \dfrac{N'_1}{N'}, \\[2ex]
\operatorname{cn} nu = (1-z)^{-\frac{n}{2}}\ \dfrac{N'_2}{N'}, \\[2ex]
\operatorname{dn} nu = (1-k^2 z)^{-\frac{n}{2}}\ \dfrac{N'_3}{N'},
\end{cases}
$$

where

$$(75)\qquad N' = \begin{vmatrix} \frac{1}{1!} D \frac{z}{s}, & \frac{1}{2!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{m!} D^m \frac{z}{s} \\ \frac{1}{2!} D^2 \frac{z}{s}, & \frac{1}{3!} D^3 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)!} D^{m+1} \frac{z}{s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{m!} D^m \frac{z}{s}, & \frac{1}{(m+1)!} D^{m+1} \frac{z}{s}, & \cdots\cdots & \frac{1}{(2m-1)!} D^{2m-1} \frac{z}{s} \end{vmatrix},$$

$$(76)\qquad N_1' = \begin{vmatrix} 0, & \frac{1}{s}, & \cdots\cdots & \frac{1}{(m-1)!} D^{m-1} \frac{1}{s} \\ \frac{1}{s}, & \frac{1}{1!} D \frac{1}{s}, & \cdots\cdots & \frac{1}{m!} D^m \frac{1}{s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m-1)!} D^{m-1} \frac{1}{s}, & \frac{1}{m!} D^m \frac{1}{s}, & \cdots\cdots & \frac{1}{(2m-1)!} D^{2m-1} \frac{1}{s} \end{vmatrix},$$

$$(77)\qquad N_2' = \begin{vmatrix} (1-z)^m, & (1-z)^{m-1}, & \cdots\cdots & 1 \\ \frac{z}{s}, & \frac{1}{1!} D \frac{z}{s}, & \cdots\cdots & \frac{1}{m!} D^m \frac{z}{s} \\ \frac{1}{1!} D \frac{z}{s}, & \frac{1}{2!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)!} D^{m+1} \frac{z}{s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m-1)!} D^{m-1} \frac{z}{s}, & \frac{1}{m!} D^m \frac{z}{s}, & \cdots\cdots & \frac{1}{(2m-1)!} D^{2m-1} \frac{z}{s} \end{vmatrix},$$

$$(78)\qquad N_3' = \begin{vmatrix} (1-k^2z)^m, & k^2(1-k^2z)^{m-1}, & \cdots\cdots & k^{2m} \\ \frac{z}{s}, & \frac{1}{1!} D \frac{z}{s}, & \cdots\cdots & \frac{1}{m!} D^m \frac{z}{s} \\ \frac{1}{1!} D \frac{z}{s}, & \frac{1}{2!} D^2 \frac{z}{s}, & \cdots\cdots & \frac{1}{(m+1)!} D^{m+1} \frac{z}{s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(m-1)!} D^{m-1} \frac{z}{s}, & \frac{1}{m!} D^m \frac{z}{s}, & \cdots\cdots & \frac{1}{(2m-1)!} D^{2m-1} \frac{z}{s} \end{vmatrix}.$$

N', N'_2, N'_3 may also be written as follows:

$$(79)\qquad N'=\begin{vmatrix} z^m, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \dots\dots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ -z^{m-1}, & \frac{1}{1!}D\frac{1}{s}, & \frac{1}{2!}D^2\frac{1}{s}, & \dots\dots & \frac{1}{m!}D^m\frac{1}{s} \\ z^{m-2}, & \frac{1}{2!}D^2\frac{1}{s}, & \frac{1}{3!}D^3\frac{1}{s}, & \dots\dots & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s} \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^m, & \frac{1}{m!}D^m\frac{1}{s}, & \frac{1}{(m+1)!}D^{m+1}\frac{1}{s}, & \dots\dots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix},$$

$$(80)\qquad N'_2=\begin{vmatrix} 0, & (1-z)^m, & (1-z)^{m-1}, & \dots\dots & 1 \\ -z^m, & 0, & \frac{1}{s}, & \dots\dots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ z^{m-1}, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \dots\dots & \frac{1}{m!}D^m\frac{1}{s} \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{m+1}, & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \dots\dots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix},$$

$$(81)\qquad N'_3=\begin{vmatrix} 0, & (1-k^2z)^m, & k^2(1-k^2z)^{m-1}, & \dots\dots & k^{2m} \\ -z^m, & 0, & \frac{1}{s}, & \dots & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s} \\ z^{m-1}, & \frac{1}{s}, & \frac{1}{1!}D\frac{1}{s}, & \dots\dots & \frac{1}{m!}D^m\frac{1}{s} \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{m+1}, & \frac{1}{(m-1)!}D^{m-1}\frac{1}{s}, & \frac{1}{m!}D^m\frac{1}{s}, & \dots\dots & \frac{1}{(2m-1)!}D^{2m-1}\frac{1}{s} \end{vmatrix}.$$

§ 12.

We are now in a position to introduce R_n and S_n into the multiplication-formulae. In some cases, we might have introduced S_n rather than R_n; but, for the sake of uniformity, we have chosen those

forms which are expressible by R_n. On introducing R_n, it will be seen that there comes out a certain power of z as a factor in the numerator as well as in the denominator, which may be cancelled against each other, leaving rational integral functions of $\sqrt{z} = \operatorname{sn} u$ both in the numerator and the denominator. The final results are:

$$n \text{ odd}, \quad m = \frac{n}{2}.$$

$$(82) \quad \operatorname{sn} nu = (-1)^m \sqrt{z} \begin{vmatrix} 1, & 2.1-z.1-k^2z, & (2.1-z.1-k^2z)^2, & \ldots & (2.1-z.1-k^2z)^{m+1} \\ 0, & 1, & R_1, & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ \cdots & \cdots & \cdots & & \cdots \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(83) \quad \operatorname{cn} nu = \sqrt{1-z} \begin{vmatrix} 1, & -2.z.1-k^2z, & (-2.z.1-k^2z)^2, & \ldots & (-2.z.1-k^2z)^{m+1} \\ 0, & 1, & R_1 & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ \cdots & \cdots & \cdots & & \cdots \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(84) \quad \operatorname{dn} nu = \sqrt{1-k^2z} \begin{vmatrix} 1, & -2.k^2z.1-z, & (-2.k^2z.1-z)^2, & \ldots & (-2.k^2z.1-z)^{m+1} \\ 0, & 1, & R_1, & \ldots & R_m \\ 1, & R_1, & R_2, & \ldots & R_{m+1} \\ R_{m-1}, & R_m, & R_{m+1}, & \ldots & R_{2m} \end{vmatrix}, \div$$

$$(85) \quad \text{denom.} = \begin{vmatrix} 1, & R_1, & \ldots R_m \\ R_1, & R_2, & \ldots R_{m+1} \\ \cdots & \cdots & \cdots \\ R_m, & R_{m+1}, & \ldots R_{2m} \end{vmatrix},$$

$$n \text{ even}, \quad m = \frac{n-1}{2},$$

$$\text{(86)} \qquad \operatorname{sn} nu = (-1)^m 2s \begin{vmatrix} 0, & 1, & \dots & R_{m-1} \\ 1, & R_1, & \dots & R_m \\ \dots & \dots & \dots & \dots \\ R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\text{(87)} \qquad \operatorname{cn} nu = \begin{vmatrix} 0, & 1, & -2.z.1-k^2z, & \dots & (-2.z.1-k^2z)^m \\ 1, & 0, & 1, & \dots & R_{m-1} \\ 2.1-z.1-k^2z, & 1, & R_1, & \dots & R_m \\ \dots & \dots & \dots & \dots & \dots \\ (2.1-z.1-k^2z)^m, & R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\text{(88)} \qquad \operatorname{dn} nu = \begin{vmatrix} 0, & 1, & -2.k^2z.1-z, & \dots & (-2.k^2z.1-z)^m \\ 1, & 0, & 1, & \dots & R_{m-1} \\ 2.1-z.1-k^2z, & 1, & R_1, & \dots & R_m \\ \dots & \dots & \dots & \dots & \dots \\ (2.1-z.1-k^2z)^m, & R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\text{(89)} \qquad \text{denom.} = -\begin{vmatrix} 1, & 1, & R_1, & \dots R_{m-1} \\ 2.1-z.1-k^2z, & R_1, & R_2, & \dots R_m \\ \dots & \dots & \dots & \dots \\ (2.1-z.1-k^2z)^m, & R_m, & R_{m+1}, & \dots R_{2m-1} \end{vmatrix}.$$

Part Second.

§ 13.

To avoid confusion, we shall adopt once for all the following notation:

$$x = \operatorname{sn} u, \quad \xi = \sqrt{k}\operatorname{sn} u, \quad z = \operatorname{sn}^2 u.$$

Following Jacobi in his lectures, we write*

ERRATA.

Page 193, line 10, *for* $\frac{\pi^2}{K^2}$ *read* $\frac{K^2}{\pi^2}$.

,, ,, , line 11, *for* $\frac{d\omega}{dK}$ *read* $\frac{d\omega}{dk}$.

,, 194, lines 8 and 11, *for* $\sqrt{k'}\operatorname{dn} nu$ *read* $\frac{1}{\sqrt{k'}}\operatorname{dn} nu$.

but

$$\frac{d\omega}{dK} = -\frac{2}{k(1-k^2)\left(\frac{2K}{\pi}\right)^2} = -\frac{\pi^2}{2kk'^2K^2},$$

hence

$$\frac{d^2\theta}{du^2} + 2kk'^2\frac{d\log K}{dk}\,u\,\frac{d\theta}{du} + 2kk'^2\frac{d\theta}{dk} = 0. \tag{90}$$

Observe that the same differential equation is satisfied by θ_1, θ_2 and θ_3.

* Jacobi's gesammelte Werke, Bd. I, pp. 501, 511, and 512.
** Ditto p. 259.
† Ditto p. 260.

$$n \text{ even}, \quad m = \frac{n-1}{2},$$

$$\text{(86)} \qquad \operatorname{sn} nu = (-1)^m 2s \begin{vmatrix} 0, & 1, & \dots & R_{m-1} \\ 1, & R_1, & \dots & R_m \\ \dots & \dots & \dots & \dots \\ R_{m-1}, & R_m, & \dots & R_{2m-1} \end{vmatrix}, \div$$

$$\begin{vmatrix} 0, & 1, & -2.z.1-k^2z, & \dots & (-2.z.1-k^2z)^m \\ 1, & 0 & 1 & \text{[illegible]} & \\ \dots & \dots & \dots & \dots & \dots \\ (2.1-z.1-k^2z)^m, & R_m, & R_{m+1}, & \dots & R_{2m-1} \end{vmatrix}.$$

Part Second.

§ 13.

To avoid confusion, we shall adopt once for all the following notation :

$$x=\operatorname{sn} u, \quad \xi=\sqrt{k}\operatorname{sn} u, \quad z=\operatorname{sn}^2 u.$$

Following Jacobi in his lectures, we write*

$$\theta(u)=1+2\sum_{n=1}^{n=\infty}(-1)^n q^{n^2}\cos\frac{n\pi u}{K}, \qquad \theta_2(u)=2\sum_{n=1}^{n=\infty} q^{\left(\frac{2n-1}{2}\right)^2}\cos\frac{(2n-1)\pi u}{2K},$$

$$\theta_1(u)=2\sum_{n=1}^{n=\infty}(-1)^{n-1} q^{\left(\frac{2n-1}{2}\right)^2}\sin\frac{(2n-1)\pi u}{2K}, \qquad \theta_3(u)=1+2\sum_{n=1}^{n=\infty} q^{n^2}\cos\frac{n\pi u}{K};$$

and then

$$\sqrt{k}\operatorname{sn} u=\frac{\theta_1(u)}{\theta(u)}, \qquad \sqrt{\frac{k}{k'}}\operatorname{cn} u=\frac{\theta_2(u)}{\theta(u)}, \qquad \frac{1}{\sqrt{k'}}\operatorname{dn} u=\frac{\theta_3(u)}{\theta(u)}.$$

Put $\omega=-\log q=\frac{\pi K'}{K}$ **, then

$$\frac{d\theta}{d\omega}=\frac{\pi^2}{K^2}\frac{d^2\theta}{du^2}-\frac{u}{K}\frac{dK}{d\omega}\frac{d\theta}{du};$$

but

$$\frac{d\omega}{dK}=-\frac{2}{k(1-k^2)\left(\frac{2K}{\pi}\right)^2}=-\frac{\pi^2}{2kk'^2K^2}\,\dagger,$$

hence

$$(90) \qquad \frac{d^2\theta}{du^2}+2kk'^2\frac{d\log K}{dk}u\frac{d\theta}{du}+2kk'^2\frac{d\theta}{dk}=0.$$

Observe that the same differential equation is satisfied by θ_1, θ_2 and θ_3.

---

* Jacobi's gesammelte Werke, Bd. I, pp. 501, 511, and 512.

** Ditto p. 259.

† Ditto p. 260.

Equation (90) may also be written in the form

$$(91)\qquad \frac{d^2\log\theta}{du^2}+\left(\frac{d\log\theta}{du}\right)^2+2kk'^2\frac{d\log K}{dk}\,u\,\frac{d\log\theta}{du}+2kk'^2\frac{d\log\theta}{dk}=0\,;$$

similarly $\dfrac{d^2\log\theta_1}{du^2}+\left(\dfrac{d\log\theta_1}{du}\right)^2+2kk'^2\dfrac{d\log K}{dk}\,u\,\dfrac{d\log\theta}{du}+2kk'^2\dfrac{d\log\theta}{dk}=0\,.$

Subtracting the former equation from the latter and replacing $\dfrac{\theta_1(u)}{\theta(u)}$ by ξ, we get

$$(92)\qquad \frac{d^2\log\xi}{du^2}+\left(\frac{d\log\xi}{du}\right)^2+2\left\{\frac{d\log\xi}{du}+kk'^2\frac{d\log K}{dk}\,u\right\}\frac{d\log\xi}{du}+2kk'^2\frac{d\log\xi}{dk}=0.$$

Now

$$\sqrt{k}\,\mathrm{sn}\,nu=\frac{\theta_1(nu)}{\theta(nu)},\quad \sqrt{\frac{k}{k'}}\,\mathrm{cn}\,nu=\frac{\theta_2(nu)}{\theta(nu)},\quad \sqrt{k'}\,\mathrm{dn}\,nu=\frac{\theta_3(nu)}{\theta(nu)}\,;$$

multiplying the numerators and denominators on the right-hand side by $\dfrac{\theta^{n^2-1}(0)}{\theta^{n^2}(u)}$,

$$\sqrt{k}\,\mathrm{sn}\,nu=\frac{V_1}{V},\quad \sqrt{\frac{k}{k'}}\,\mathrm{cn}\,nu=\frac{V_2}{V},\quad \sqrt{k'}\,\mathrm{dn}\,nu=\frac{V_3}{V},$$

where

$$(93)\qquad \left\{\begin{aligned} V&=\frac{\theta(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}, & V_2&=\frac{\theta_2(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)},\\ V_1&=\frac{\theta_1(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}, & V_3&=\frac{\theta_3(nu)\theta^{n^2-1}(0)}{\theta^{n^2}(u)}.\end{aligned}\right.$$

Put nu instead of u in equation (91), thus:—

$$\frac{d^2\log\theta(nu)}{du^2}+\left(\frac{d\log\theta(nu)}{du}\right)^2+2u^2kk'^2\frac{d\log K}{dk}\,u\,\frac{d\log\theta(nu)}{du}+2u^2kk'^2\frac{d\log\theta(nu)}{dk}=0\,.$$

Now

$$\log \theta(nu) = \log V + n^2 \log \theta(u) - (n^2-1) \log \theta(0) .$$

Substituting in (91), we get

$$(94)\quad \frac{d^2\log V}{du^2} + \left(\frac{d\log V}{du}\right)^2 + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{d\log V}{du}$$
$$+ 2n^2kk'^2\frac{d\log V}{dk} - n^2(n^2-1)\left\{\frac{d^2\log\theta(u)}{du^2} + 2kk'^2\frac{d\log\theta(0)}{dk}\right\} = 0.$$

Again, $\quad \dfrac{d\log\theta(u)}{du} = Z(u)\,, \qquad \theta(0) = \sqrt{\dfrac{2k'K}{\pi}}$ *;

differentiating, and observing $\dfrac{dK}{dk} = \dfrac{E - k'^2K}{kk'^2}$, we obtain

$$(95)\qquad \frac{d^2\log\theta(u)}{du^2} + 2kk'^2\frac{d\log\theta(0)}{dk} = -k^2\text{sn}^2u.$$

Substituting this in (94),

$$(96)\quad \frac{d^2\log V}{du^2} + \left(\frac{d\log V}{du}\right)^2 + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{d\log V}{du}$$
$$+ 2n^2kk'^2\frac{d\log V}{dk} + n^2(n^2-1)k^2\text{sn}^2u = 0\,,$$

or

$$(97)\qquad \frac{d^2V}{du^2} + 2n^2\left\{\frac{d\log\theta(u)}{du} + kk'^2\frac{d\log K}{dk}u\right\}\frac{dV}{du}$$
$$+ 2n^2kk'^2\frac{dV}{dk} + n^2(n^2-1)k^2\text{sn}^2u\,V = 0.$$

Introducing $\xi = \sqrt{k}\,\text{sn}\,u$ as the new independent variable, and observing (92),

* Jacobi's gesammelte Werke, Bd. I, pp. 198 and 235.

$$\left(\frac{d\xi}{du}\right)^2\frac{d^2V}{d\xi^2}-(n^2-1)\frac{d^2\xi}{du^2}\frac{dV}{d\xi}+2n^2kk'^2\frac{dV}{dk}+n^2(n^2-1)k\xi^2V=0\,.$$

Since

$$\left(\frac{d\xi}{du}\right)^2=k\left\{1-\left(k+\frac{1}{k}\right)\xi^2+\xi^4\right\}\,,\ \frac{d^2\xi}{du^2}=k\left\{-\left(k+\frac{1}{k}\right)\xi+2\xi^3\right\}\,,$$

equation (97) takes the form

$$(98)\quad \left\{1-\left(k+\frac{1}{k}\right)\xi^2+\xi^4\right\}\frac{d^2V}{d\xi^2}-(n^2-1)\left\{-\left(k+\frac{1}{k}\right)\xi+2\xi^3\right\}\frac{dV}{d\xi}$$
$$+2n^2k'^2\frac{dV}{dk}+n^2(n^2-1)\xi^2V=0$$

This equation is also satisfied by V_1, V_2, and V_3.

With Jacobi, we may put $k+\dfrac{1}{k}=\alpha$ and then equation (98) becomes

$$(99)\quad (1-\alpha\xi^2+\xi^4)\frac{d^2V}{d\xi^2}-(n^2-1)(-\alpha\xi+2\xi^3)\frac{dV}{d\xi}+2(4-\alpha^2)n^2\frac{dV}{d\alpha}$$
$$+n^2(n^2-1)\xi^2V=0\,,$$

in which form, the partial differential equation was given for the first time by Jacobi.

The above demonstration of Jacobi's partial differential equation is substantially the same as the one given by Briot and Bouquet* in their well-known treatise on elliptic functions, the only difference being that, in their demonstration, the intermediary steps are conducted by means of Weierstrass' function Al.

Briot and Bouquet give also the following forms of the partial differential equation :—

* Loc. cit. p. 529.

$$(100)\quad \left(1 - \frac{1-2k^2}{kk'}\eta^2 - \eta^4\right)\frac{d^2V}{d\eta^2} + (n^2-1)\left(\frac{1-2k^2}{kk'}\eta + 2\eta^3\right)\frac{dV}{d\eta}$$
$$+ 2n^2k'\frac{dV}{dk} + n^2(n^2-1)\left(\frac{k}{k'} - \eta^2\right)V = 0\,,$$

$$(101)\quad \left(1 - \frac{2-k^2}{k'}\zeta^2 + \zeta^4\right)\frac{d^2V}{d\zeta^2} + (n^2-1)\left(\frac{2-k^2}{k'}\zeta - 2\zeta^3\right)\frac{dV}{d\zeta}$$
$$- 2n^2kk'\frac{dV}{dk} - n^2(n^2-1)\left(\frac{1}{k'} - \zeta^2\right)V = 0\,,$$

where $\eta = \sqrt{\frac{k}{k'}}\operatorname{cn} u$ and $\zeta = \frac{1}{\sqrt{k'}}\operatorname{dn} u$.

§ 11.

We shall have, by suitably determining the constants,

$$(102)\quad \begin{cases} V_1 = \sqrt{k}\,x\,A(x^2), & A(0) = n, \\ V_2 = \sqrt{\frac{k}{k'}}\sqrt{1-x^2}\,B(x^2), & B(0) = 1, \\ V_3 = \frac{1}{\sqrt{k'}}\sqrt{1-k^2x^2}\,C(x^2), & C(0) = 1, \\ V\;\, = D(x^2), & D(0) = 1, \end{cases} \qquad n \text{ odd,}$$

$$(103)\quad \begin{cases} V_1 = \sqrt{k}\,x\sqrt{1-x^2}\sqrt{1-k^2x^2}\,A(x^2), & A(0) = n, \\ V_2 = \sqrt{\frac{k}{k'}}\,B(x^2), & B(0) = 1, \\ V_3 = \frac{1}{\sqrt{k'}}\,C(x^2), & C(0) = 1, \\ V\;\, = D(x^2), & D(0) = 1, \end{cases} \qquad n \text{ even,}$$

where A, B, C, D are rational integral functions of x^2.

By division, we obtain

$$
\begin{array}{llll}
 & \quad n \text{ odd,} & & \quad n \text{ even,} \\[6pt]
 & \operatorname{sn} nu = \dfrac{xA(x^2)}{D(x^2)}, & & \operatorname{sn} nu = \dfrac{x\sqrt{1-x^2}\sqrt{1-k^2x^2}A(x^2)}{D(x^2)}, \\[12pt]
(104) & \operatorname{cn} nu = \dfrac{\sqrt{1-x^2}B(x^2)}{D(x^2)}, & & \operatorname{cn} nu = \dfrac{B(x^2)}{D(x^2)}, \\[12pt]
 & \operatorname{dn} nu = \dfrac{\sqrt{1-k^2x^2}C(x^2)}{D(x^2)}, & & \operatorname{dn} nu = \dfrac{C(x^2)}{D(x^2)}.
\end{array}
$$

When n is odd, A, B, C, D are all of the degree n^2-1 in x, and when n is even, A is of the degree n^2-4 while A, B, C are all of the degree n^2 in x. Moreover, A, B, C, D are rational integral functions of k^2 whose coefficients are integral numbers. The coefficients of the highest power of x in A, B, C, D are respectively

$$(-1)^{\frac{n-1}{2}}\, k^{\frac{n^2-1}{2}}, \qquad k^{\frac{n^2-1}{2}}, \qquad k^{\frac{n^2-1}{2}}, \qquad (-1)^{\frac{n-1}{2}}\, n\, k^{\frac{n^2-1}{2}},$$

or

$$(-1)^{\frac{n-2}{2}}\, n\, k^{\frac{n^2-4}{2}}, \qquad k^{\frac{n^2}{2}}, \qquad k^{\frac{n^2}{2}}, \qquad (-1)^{\frac{n}{2}}\, k^{\frac{n^2}{2}},$$

according as n is odd or even.

Write

$$
(105) \qquad
\begin{aligned}
A &= \Sigma A_{2m} x^{2m} = \Sigma A'_{2m}(1-x^2)^m = \Sigma A''_{2m}(1-k^2x^2)^m, \\
B &= \Sigma B_{2m} x^{2m} = \Sigma B'_{2m}(1-x^2)^m = \Sigma B''_{2m}(1-k^2x^2)^m, \\
C &= \Sigma C_{2m} x^{2m} = \Sigma C'_{2m}(1-x^2)^m = \Sigma C''_{2m}(1-k^2x^2)^m, \\
D &= \Sigma D_{2m} x^{2m} = \Sigma D'_{2m}(1-x^2)^m = \Sigma D''_{2m}(1-k^2x^2)^m;
\end{aligned}
$$

then, we have the well-known relations*

* Compare the work of Briot et Bouquet already referred to, or Baehr, *Sur les formules pour la multiplication des fonctions elliptiques de la première espèce*, Grunert's Archiv der Mathematik, Bd. XXXVI, pp. 125-176.

$$A_{2m}(k)=k^{2m}A_{2m}\left(\frac{1}{k}\right),\quad A''_{2m}(k)=A'_{2m}\left(\frac{1}{k}\right),$$

(106) $$C_{2m}(k)=k^{2m}B_{2m}\left(\frac{1}{k}\right),\quad C''_{2m}(k)=B'_{2m}\left(\frac{1}{k}\right),\quad B''_{2m}(k)=C'_{2m}\left(\frac{1}{k}\right),$$

$$D_{2m}(k)=k^{2m}D_{2m}\left(\frac{1}{k}\right),\quad D''_{2m}(k)=D'_{2m}\left(\frac{1}{k}\right).$$

Observe that A_{2m}, B_{2m}, C_{2m}, D_{2m}, A'_{2m}, B'_{2m}, C'_{2m}, D'_{2m} are integral functions of k^2 of the degree at most equal to m.

§ 15.

Consider first the case where n is an odd number, and put $n^2-1=4p=8q$.

Introducing the new variable $\xi=\sqrt{k}\,x$ in

$$\operatorname{sn} nu=\frac{x\,\Sigma' A_{2m}\,x^{2m}}{\Sigma' D_{2m}\,x^{2m}},$$

we get

(107) $$\sqrt{k}\operatorname{sn} nu=\frac{\xi\,\Sigma' a_{2m}\,\xi^{2m}}{\Sigma' d_{2m}\,\xi^{2m}},$$

where a_{2m}, d_{2m} are rational integral functions of $\alpha=k+\dfrac{1}{k}$.

We may suppose A and D to be arranged according to the powers of α, thus:—

(108)
$$A=\Sigma' A_{2m}x^{2m}=\Sigma' a_{2m}\xi^{2m}=\Sigma' E_m\alpha^m,$$
$$D=\Sigma' D_{2m}x^{2m}=\Sigma' d_{2m}\xi^{2m}=\Sigma' H_m\alpha^m.$$

From the well known relations

$$(109)\quad A(x,k)=(-1)^{\frac{n-1}{2}}D\left(\frac{1}{kx},k\right)k^{2p}x^{4p},\quad D(x,k)=(-1)^{\frac{n-1}{2}}A\left(\frac{1}{kx},k\right)k^{2p}x^{4p},$$

which may also be written in the form

$$(110)\quad A(\alpha,\xi)=(-1)^{\frac{n-1}{2}}D\left(\alpha,\frac{1}{\xi}\right)\xi^{4p},\qquad D(\alpha,\xi)=(-1)^{\frac{n-1}{2}}A\left(\alpha,\frac{1}{\xi}\right)\xi^{4p},$$

we see that A and D are of the same degree in α and that, moreover,

$$(111)\qquad E_m(\xi)=(-1)^{\frac{n-1}{2}}H_m\left(\frac{1}{\xi}\right)\xi^{4p},\quad H_m(\xi)=(-1)^{\frac{n-1}{2}}E_m\left(\frac{1}{\xi}\right)\xi^{4p}.$$

When the multiplicator n is required to be put in evidence, we shall write $A[n]$ and $D[n]$. Now the terms containing the highest power of α in $A[3]$, $A[5]$, $A[7]$ are $-2^2\xi^2\alpha$, $2^6\xi^{14}\alpha^3$, $-2^{12}\xi^{20}\alpha^6$, and those in $D[3]$, $D[5]$, $D[7]$ are $2^2\xi^6\alpha$, $2^6\xi^{10}\alpha^3$, $2^{12}\xi^{28}\alpha^6$. By virtue of the relation

$$(112)\qquad A[n+2]A[n-2]=D^2[2]A^2[n]-(1-\alpha\xi^2+\xi^4)A^2[2]D^2[n],$$

which is easily deducible from the addition-equation, we conclude by applying mathematical induction that, generally, the terms involving the highest power of α in $A[n]$, $D[n]$ are

$$(113)\qquad E_q\alpha^q=(-1)^{\frac{n-1}{2}}2^p\xi^{\lambda_n}\alpha^q,\quad H_q\alpha^q=2^p\xi^{\mu_n}\alpha^q,$$

where $\lambda_n+\mu_n=n^2-1$.

To find λ_n, μ_n, we deduce from (112)

$$\lambda_{n+2}+\lambda_{n-2}=2\mu_n+2=2n^2-2\lambda_n,$$

which may be written

$$\left\{\lambda_{n+2}-\frac{(n+2)^2}{2}+1\right\}+2\left\{\lambda_n-\frac{n^2}{2}+1\right\}+\left\{\lambda_{n-2}-\frac{(n-2)^2}{2}+1\right\}=0\,,$$

or, putting for a moment $\left(\lambda_n - \frac{n^2}{2} + 1\right) \equiv \psi_n$,

$$\psi_{n+2} + 2\psi_n + \psi_{n-2} = 0 .$$

Observing $\psi_1 = \frac{1}{2}$, $\psi_3 = -\frac{3}{2}$, we find

$$\psi_n = \text{ coef. of } x^n \text{ in } \frac{1}{2}\,\frac{x - x^3}{(1+x^2)^2} = (-1)^{\frac{n-1}{2}}\frac{n}{2} .$$

Hence,

$$(114) \qquad \lambda_n = \frac{n^2}{2} + (-1)^{\frac{n-1}{2}}\frac{n}{2} - 1 , \quad \mu_n = \frac{n^2}{2} - (-1)^{\frac{n-1}{2}}\frac{n}{2} ,$$

that is,

$$\lambda_n = \frac{1}{2}n(n-1) - 1 , \quad \mu_n = \frac{1}{2}n(n+1) ,$$

or

$$\lambda_n = \frac{1}{2}n(n+1) - 1 , \quad \mu_n = \frac{1}{2}n(n-1) ,$$

according as $\frac{n-1}{2}$ is odd or even.

This premised, we now substitute $D = \sum_{m=0}^{m=q} H_m \alpha^m$ in (99), which may be written in the form

$$\alpha\xi^2\frac{d^2V}{d\xi^2} - (n^2-1)\alpha\xi\frac{dV}{d\xi} + 2n^2\alpha^2\frac{dV}{d\alpha} = (1+\xi^4)\frac{d^2V}{d\xi^2} - 2(n^2-1)\xi^3\frac{dV}{d\xi}$$
$$+ n^2(n^2-1)\xi^2 V + 8n^2\frac{dV}{d\alpha} ,$$

and obtain

$$(115) \quad \xi^2\frac{d^2H_q}{d\xi^2} - (n^2-1)\xi\frac{dH_q}{d\xi} + 2n^2qH_q = 0 ,$$

$$(116) \quad \xi^2\frac{d^2H_{q-1}}{d\xi^2} - (n^2-1)\xi\frac{dH_{q-1}}{d\xi} + 2n^2(q-1)H_{q-1} = (1+\xi^4)\frac{d^2H_q}{d\xi^2} - 2(n^2-1)\xi^3\frac{dH_q}{d\xi}$$
$$+ n^2(n^2-1)\xi^2H_q .$$

and, generally,

$$(117)\qquad \xi^2\frac{d^2H_m}{d\xi^2}-(n^2-1)\xi\frac{dH_m}{d\xi}+2n^2mH_m=U_m\,,\quad m=q-2,\ q-3,\ \dots 0\,,$$

where $$U_m=(1+\xi^4)\frac{d^2H_{m+1}}{d\xi^2}-2(n^2-1)\xi^3\frac{dH_{m+1}}{d\xi}+n^2(n^2-1)\xi^2H_{m+1}$$
$$+8n^2(m+2)H_{m+2}\,.$$

It will be seen that equation (115) is satisfied by ξ^{μ_n}; indeed we might have determined μ_n by means of this equation, for if the operator $\xi\dfrac{d}{d\xi}$ be denoted by ϑ, then the differential equation may be written

$$(\vartheta^2-n^2\vartheta+2n^2q)\,H_q=0\,,$$

the general solution of which is

$$C'\xi^{\frac{1}{2}n(n-1)}+C''\xi^{\frac{1}{2}n(n+1)},$$

where C', C'' are arbitrary constants; and, as H_q can contain only even powers of ξ, $C'=0$, or $C''=0$, according as $\dfrac{n-1}{2}$ is odd or even.

Let us now investigate equation (116). For this purpose, it will be convenient to distinguish two cases according as $\dfrac{n-1}{2}$ is even or odd. When $\dfrac{n-1}{2}$ is even, $\mu_n=\dfrac{1}{2}n(n-1)$ and $H_q=2^p\xi^{\frac{1}{2}n(n-1)}$, further

$$(1+\xi^4)\frac{d^2H_q}{d\xi^2}-2(n^2-1)\xi^3\frac{dH_q}{d\xi}+n^2(n^2-1)\xi^2H_q$$
$$=2^{p-2}\{(n-2)(n-1)n(n+1)\xi^{\frac{1}{2}n(n-1)-2}+(n-1)n(n+1)(n+2)\xi^{\frac{1}{2}n(n-1)+2}\}\,,$$

so that equation (116) now assumes the form

$$(\vartheta^2-n^2\vartheta+2n^2(q-1))H_{q-1}$$
$$=2^{p-2}\{(n-2)(n-1)n(n+1)\xi^{\frac{1}{2}n(n-1)-2}+(n-1)n(n+1)(n+2)\xi^{\frac{1}{2}n(n-1)+2}\}\,.$$

Now, q being equal to $\frac{n^2-1}{8}$,

$$\vartheta^2-n^2\vartheta+2n^2(q-1)=\left(\vartheta-\frac{n^2}{2}\right)^2-\left(\frac{3n}{2}\right)^2.$$

The complementary function is thus

$$C'\xi^{\frac{1}{2}n(n+3)}+C''\xi^{\frac{1}{2}n(n-3)};$$

but, as $\frac{1}{2}n(n-3)$ is odd and H_{q-1} contains only even powers of ξ, $C''=0$. Again, a particular integral is $-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-2}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+2}$, which, together with the complementary function just found, gives

$$H_{q-1}=-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-2}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+2}+\Gamma_1\xi^{\frac{1}{2}n(n+3)},$$

where Γ_1 is an as-yet undetermined function of n.

When $\frac{n-1}{2}$ is odd, we find likewise

$$H_{q-1}=\Gamma_1\xi^{\frac{1}{2}n(n-3)}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)}-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+1)}.$$

By virtue of (111),

$$E_{q-1}=\Gamma_1\xi^{\frac{1}{2}n(n-3)-1}-2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n+1)-3}-2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n+1)+1},\quad \frac{n-1}{2}\text{ even},$$

$$E_{q-1}=2^{p-3}(n-1)n\xi^{\frac{1}{2}n(n-1)-3}+2^{p-3}n(n+1)\xi^{\frac{1}{2}n(n-1)+1}-\Gamma_1\xi^{\frac{1}{2}n(n+3)-1},\quad \frac{n-1}{2}\text{ odd}.$$

We may put the multiplicator n in evidence by writing

$$A[n]=\Sigma\overset{n}{E}_m\alpha^m,\quad D[n]=\Sigma\overset{n}{H}_m\alpha^m,\quad q\equiv q_n,\quad p\equiv p_n,$$

and then equation (112) may be written

$$(118)\quad \Sigma\overset{n+2}{E}_m\alpha^m.\,\Sigma\overset{n-2}{E}_m\alpha^m=(1-\xi^4)^2\left[\Sigma\overset{n}{E}_m\alpha^m\right]^2-4(1-\alpha\xi^2+\xi^4)\left[\Sigma\overset{n}{H}_m\alpha^m\right]^2,$$

whence, equating the coefficients of the second highest power of α,

$$(119)\quad \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1}+\overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1}=(1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2-4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2$$
$$+8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1}.$$

If $\frac{n-1}{2}$ be odd, so that $\frac{n\pm 2-1}{2}$ is even, then the term containing the lowest power of ξ

$$
\begin{array}{llll}
\text{in} & \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1} & \text{is} & 2^{p_{n+2}}\overset{n-2}{\Gamma_1}\xi^{n^2-n+6}, \\
,, & \overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1} & ,, & 2^{p_{n-2}}\overset{n+2}{\Gamma_1}\xi^{n^2-n-2}, \\
,, & (1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2 & ,, & 2^{2p_n}\xi^{n^2-n-2}, \\
,, & -4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2 & ,, & -2^{2p_n+2}\xi^{n^2+n}, \\
,, & 8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_{n-1}} & ,, & 2^{p_n+3}\overset{n}{\Gamma_1}\xi^{n^2-n+2}.
\end{array}
$$

Equating the coefficients of the lowest power of ξ (that is ξ^{n^2-n-2}) on the two sides of equation (119), we get

$$\overset{n+2}{\Gamma_1}=2^{2p_n-p_{n-2}}=2^{\frac{(n-1)(n+5)}{4}},\qquad \frac{n-1}{2}\ \text{odd},$$

or, writing n instead of $n+2$,

$$\overset{n}{\Gamma_1}=2^{\frac{(n-3)(n+3)}{4}}=2^{p_n-2}=2^{2(q_n-1)},\qquad \frac{n-1}{2}\ \text{even}.$$

Again, when $\frac{n-1}{2}$ is even, so that $\frac{n\pm 2-1}{2}$ is odd, the term containing the highest power of ξ

$$
\begin{array}{llll}
\text{in} & \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-1} & \text{is} & 2^{p_{n+2}}\overset{n-2}{\Gamma_1}\xi^{n^2+n-2}, \\
,, & \overset{n-2}{E}_{q_{n-2}}\overset{n+2}{E}_{q_{n+2}-1} & ,, & 2^{p_{n-2}}\overset{n+2}{\Gamma_1}\xi^{n^2+n+6}, \\
,, & (1-\xi^4)^2\left[\overset{n}{E}_{q_n}\right]^2 & ,, & 2^{2p_n}\xi^{n^2+n+6}, \\
,, & -4(1+\xi^4)\left[\overset{n}{H}_{q_n}\right]^2 & ,, & -2^{2p_n+2}\xi^{n^2-n+4}, \\
,, & 8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_{n-1}} & ,, & 2^{p_n+3}\xi^{n^2+n+2}.
\end{array}
$$

Equating the coefficients of the highest power of ξ (that is ξ^{n^2+n+6}) on the two sides of equation (119), we obtain

$$P_1^{\,n+2} = 2^{2p_n - p_{n-2}} = 2^{\frac{(n-1)(n+5)}{4}}, \qquad \frac{n-1}{2} \text{ odd},$$

or, writing n instead of $n+2$,

$$P_1^{\,n} = 2^{\frac{(n-3)(n+3)}{4}} = 2^{p_n-2} = 2^{2(q_n-1)}, \qquad \frac{n-1}{2} \text{ even}.$$

Thus we find, whether $\frac{n-1}{2}$ is odd or even,

(120) $$P_1^{\,n} = 2^{p_n-2} = 2^{2(q_n-1)}$$

and thence, when $\frac{n-1}{2}$ is even,

(121) $$H_{q-1} = -2^{p-3}(n-1)n\xi^{\frac12 n(n-1)-2} - 2^{p-3}n(n+1)\xi^{\frac12 n(n-1)+2} + 2^{p-2}\xi^{\frac12 n(n+3)},$$

(122) $$E_{q-1} = 2^{p-2}\xi^{\frac12 n(n-3)-1} - 2^{p-3}n(n+1)\xi^{\frac12 n(n+1)-3} - 2^{p-3}(n-1)n\xi^{\frac12 n(n+2)+1},$$

and, when $\frac{n-1}{2}$ is odd,

(123) $$H_{q-1} = 2^{p-2}\xi^{\frac12 n(n-3)} - 2^{p-3}n(n+1)\xi^{\frac12 n(n+1)-2} - 2^{p-3}(n-1)n\xi^{\frac12 n(n+1)+2},$$

(124) $$E_{q-1} = 2^{p-3}(n-1)n\xi^{\frac12 n(n-1)-3} + 2^{p-3}n(n+1)\xi^{\frac12 n(n-1)+1} - 2^{p-2}\xi^{\frac12 n(n+3)-1}.$$

Consider next H_{q-2}, whereby we suppose $\frac{n-1}{2}$ to be even. H_{q-2} satisfies differential equation (117)

$$\{\vartheta^2 - n^2\vartheta + 2n^2(q-2)\}H_{q-2} = U_{q-2},$$

where

$$U_{q-2} = -2^{p-5}(n-1)n(n^2-n-4)(n^2-n-6)\xi^{\frac12 n(n-1)-4} - 2^{p-4}n^2(n^2-2)(n^2-9)\xi^{\frac12 n(n-1)}$$
$$-2^{p-5}n(n+1)(n^2+n-4)(n^2+n-6)\xi^{\frac12 n(n-1)+4} + 2^{p-4}n(n+3)(n^2+3n-2)\xi^{\frac12 n(n+3)-2}$$
$$+2^{p-4}n(n-3)(n^2-3n-2)\xi^{\frac12 n(n+3)+2}.$$

The complementary function is

$$C'\xi^{\frac{n^2-\sqrt{17}\,n}{2}}+C''\xi^{\frac{n^2+\sqrt{17}\,n}{2}},$$

and, as H_{q-2} can not contain irrational powers of ξ, we must have $C'=0$, $C''=0$. Hence, $\frac{n-1}{2}$ being even,

$$(125)\quad \begin{aligned} H_{q-2}= {} & 2^{p-7}(n-1)n(n^2-n-6)\xi^{\frac{1}{2}n(n-1)-4}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n-1)} \\ & +2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n-1)+4}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n+3)-2} \\ & -2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n+3)+2} \end{aligned}$$

and thence,

$$(126)\quad \begin{aligned} E_{q-2}= {} & -2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n-3)-3}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n-3)+1} \\ & +2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n+1)-5}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n+1)-1} \\ & +2^{p-7}(n-1)n(n^2-n-6)\xi^{\frac{1}{2}n(n+1)+3}. \end{aligned}$$

When $\frac{n-1}{2}$ is odd, we find likewise (or, more simply, by writing $-n$ in place of n in (125) and (126).)

$$(127)\quad \begin{aligned} H_{q-2}= {} & -2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n-3)-2}-2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n-3)+2} \\ & +2^{p-7}n(n+1)(n^2+n-6)\xi^{\frac{1}{2}n(n+1)-4}+2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n+1)} \\ & +2^{p-7}n(n-1)(n^2-n-6)\xi^{\frac{1}{2}n(n+1)+4}, \end{aligned}$$

$$(128)\quad \begin{aligned} E_{q-2}= {} & -2^{p-7}n(n-1)(n^2-n-6)\xi^{\frac{1}{2}n(n-1)-5}-2^{p-6}(n^2-2)(n^2-9)\xi^{\frac{1}{2}n(n-1)-1} \\ & -2^{p-7}n(n-1)(n^2+n-6)\xi^{\frac{1}{2}n(n-1)+3}+2^{p-5}n(n+3)\xi^{\frac{1}{2}n(n+3)-3} \\ & +2^{p-5}n(n-3)\xi^{\frac{1}{2}n(n+3)+1}. \end{aligned}$$

Next, consider H_{q-3} which satisfies the differential equation

$$\left(\vartheta-\tfrac{1}{2}n(n+5)\right)\left(\vartheta-\tfrac{1}{2}n(n-5)\right)\ H_{q-3}=U_{q-3},$$

where

$$U_{q-3}=\frac{d^2H_{q-2}}{d\xi^2}+\xi^2(\vartheta-n^2)(\vartheta-n^2+1)H_{q-2}+\frac{n^2(n-3)(n+3)}{4}H_{q-1}.$$

When $\frac{n-1}{2}$ is even,

$$
\begin{aligned}
U_{q-3} = {} & 2^{p-9}(n-1)n(n-3)(n+2)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-6} \\
& + 2^{p-9}(n-1)n(n-3)(3n+2)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-2} \\
& + 2^{p-9}n(n+1)(n+3)(3n-2)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+2} \\
& + 2^{p-9}n(n+1)(n+3)(n-2)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+6} \\
& - 2^{p-7}n(n+3)(n+4)(n-1)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-4} \\
& - 2^{p-6}n^2(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)} \\
& - 2^{p-7}n(n-3)(n-4)(n+1)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+4}.
\end{aligned}
$$

A particular integral may easily be found. The complementary function is $\Gamma_3\xi^{\frac{1}{2}n(n-5)}$, where Γ_3 is a function of n. Hence we get, $\frac{n-1}{2}$ being even,

$$
\begin{aligned}
(129)\quad H_{q-3} = {} & \Gamma'_3\xi^{\frac{1}{2}n(n-5)} \\
& - 2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-6} \\
& - 2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-2} \\
& - 2^{p-10}n(n+3)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+2} \\
& - 2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+6} \\
& + 2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-4} \\
& + 2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)} \\
& + 2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+4},
\end{aligned}
$$

$$
\begin{aligned}
(130)\quad E_{q-3} = {} & 2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n-3)-5} \\
& + 2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n-3)-1} \\
& + 2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n-3)+3} \\
& - 2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n+1)-7} \\
& - 2^{p-10}n(n+3)(n^4-2n^3-21n^2+42n+60)\xi^{\frac{1}{2}n(n+1)-3} \\
& - 2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n+1)+1} \\
& - 2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n+1)+5} \\
& + \Gamma'_3\xi^{\frac{1}{2}n(n+5)-1}.
\end{aligned}
$$

When $\frac{n-1}{2}$ is odd, we get, by writing $-n$ instead of n in (129),

$$
\begin{aligned}
(131)\quad H_{q-3}= {} & 2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n-3)-4} \\
& +2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n-3)} \\
& +2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n-3)+4} \\
& -2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n+1)-6} \\
& -2^{p-10}n(n+3)(n^4-2n^3+21n^2+42n+60)\xi^{\frac{1}{2}n(n+1)-2} \\
& -2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n+1)+2} \\
& -2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n+1)+6} \\
& +\Gamma_3\xi^{\frac{1}{2}n(n+5)},
\end{aligned}
$$

and thence, by virtue of (111),

$$
\begin{aligned}
(132)\quad E_{q-3}= {} & -\Gamma_3\xi^{\frac{1}{2}n(n-5)-1} \\
& +2^{p-10}3^{-1}n(n-1)(n^2-n-8)(n^2-n-10)\xi^{\frac{1}{2}n(n-1)-7} \\
& +2^{p-10}n(n-3)(n^4+2n^3-21n^2-42n+60)\xi^{\frac{1}{2}n(n-1)-3} \\
& +2^{p-10}n(n+3)(n^4-2n^3+21n^2+42n+60)\xi^{\frac{1}{2}n(n-1)+1} \\
& +2^{p-10}3^{-1}n(n+1)(n^2+n-8)(n^2+n-10)\xi^{\frac{1}{2}n(n-1)+5} \\
& -2^{p-9}n(n+3)(n^2+3n-6)\xi^{\frac{1}{2}n(n+3)-5} \\
& -2^{p-8}(n^4-19n^2+98)\xi^{\frac{1}{2}n(n+3)-1} \\
& -2^{p-9}n(n-3)(n^2-3n-6)\xi^{\frac{1}{2}n(n+3)+3}.
\end{aligned}
$$

Equating the coefficients of $\alpha^{\frac{n^2-9}{4}}$ on the two sides of (118), we obtain

$$
\begin{aligned}
& \overset{n+2}{E}_{q_{n+2}}\overset{n-2}{E}_{q_{n-2}-3}+\overset{n+2}{E}_{q_{n+2}-1}\overset{n-2}{E}_{q_{n-2}-2}+\overset{n+2}{E}_{q_{n+2}-2}\overset{n-2}{E}_{q_{n-2}-1}+\overset{n+2}{E}_{q_{n+2}-3}\overset{n-2}{E}_{q_{n-2}} \\
& \qquad =(1-\xi^4)^2\left(2\overset{n}{E}_{q_n}\overset{n}{E}_{q_n-2}+\overset{n}{E}_{q_n-1}\overset{n}{E}_{q_n-1}\right) \\
& \qquad\qquad -4(1+\xi^4)\left(2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-2}+\overset{n}{H}_{q_n-1}\overset{n}{H}_{q_n-1}\right)+8\xi^2\overset{n}{H}_{q_n}\overset{n}{H}_{q_n-1}.
\end{aligned}
$$

When $\frac{n-1}{2}$ is even, the lowest power of ξ in the above equation is ξ^{n^2-3n-2}. Equating the coefficients of this power of ξ on the two sides, we get $\overset{n+2}{\Gamma}_3\,2^{p_n-2}=2^{2(p_n-2)}$, whence

$$\overset{n+2}{\Gamma_3} = 2^{\frac{(n+2)^2-1}{4}-6}, \quad \frac{n-1}{2} \text{ even}$$

or, writing n in place of $n+2$,

$$\overset{n}{\Gamma_3} = 2^{\frac{n^2-1}{4}-6} = 2^{2(q-3)}, \quad \frac{n-1}{2} \text{ odd.}$$

In like manner, we may shew that $\overset{n}{\Gamma_3}$ has the same value in the case where $\frac{n-1}{2}$ is even. Thus, whether n be odd or even,

$$(133) \qquad \Gamma_3 = 2^{2(q-3)}.$$

In determining $H_{q-\mu}$, every time $8\mu+1$ is of the form r^2, where r denotes an integer, there comes in the term $\overset{n}{\Gamma_\mu}\xi^{\frac{1}{2}n(n\pm r)}$. Indeed all the terms of H and E may readily be expressed in terms of Γ. On the other hand, Γ_μ may be determined by means of (118), as has been actually done in the cases $\mu=1$, 3; but when μ is a large number, this becomes very laborious. Now in the series

$$\Gamma_0, \ \Gamma_1, \ \Gamma_3, \ \Gamma_6, \ \Gamma_{10}, \ \Gamma_{15}, \ \Gamma_{21}, \ldots\ldots,$$

$$\Gamma_0 = 2^{2q}, \quad \Gamma_1 = 2^{2(q-1)}, \quad \Gamma_3 = 2^{2(q-3)},$$

so that most probably $\Gamma_\mu = 2^{2(q-\mu)}$; but I have not thus far succeeded in proving this generally.

For a given particular value of n, the constants Γ may be determined in a different manner. For example, take the case $n=9$.

$$\text{Sin } 9x = x(9-120x^2+432x^4-576x^6+256x^8).$$

Again $\overset{n}{E}_{q-6}$ contains the term $\overset{n}{\Gamma_6}\xi^{\frac{1}{2}n(n-7)-1}$ and $\overset{n}{H}_{q-10}$ the term $\overset{n}{\Gamma}_{10}\xi^{\frac{1}{2}n(n-9)}$. If we put $k=0$, α becomes infinite but in the same manner as $\frac{1}{k}$. Thus $\sin 9x$ contains the term $\overset{9}{\Gamma_6}x^9$, whence follows $\overset{9}{\Gamma_6} = 256$. Obviously $\overset{9}{\Gamma}_{10}=1$.

The elliptic functions of nu for $n = 2, 3, 4, 5, 6, 7, 8$ have been calculated by Baehr and others* by the primitive method of successively applying the addition-equation. Having found the values of $\overset{9}{V}$, sn $9u$ may be calculated by the above method without much difficulty.

§ 16.

As regards sn nu, n being odd, the analysis contained in the preceding section, whereby the variables are taken to be $\sqrt{k}$ sn u and α, leaves nothing to be desired; yet for the other functions, this is not the case, and it is better to have as the variables sn u and k. For the sake of uniformity, therefore, we shall once more investigate sn nu, but this time consider it as a function of sn u and k.

Changing the variable from ξ to x, equation (98) takes the form

$$(134)\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2V}{dx^2}+\{((2n^2-1)k^2-1)x-2(n^2-1)k^2x^3\}\frac{dV}{dx}$$
$$+2n^2k(1-k^2)\frac{dV}{dk}+n^2(n^2-1)k^2x^2\;V=0\,,$$

and the equation is satisfied by the numerators and denominator of $\sqrt{k}$ sn nu, $\sqrt{\frac{k}{k'}}$ cn nu, $\frac{1}{\sqrt{k'}}$ dn nu.

The numerator of sn nu, that is, $xA(x^2)$, satisfies the differential equation

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* See the paper of Baehr already referred to, and Proceedings of the Royal Society of London, Vol. XXXIII (1882) pp. 480-489.

$$(135)\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2V}{dx^2}+\{((2n^2-1)k^2-1)x-2(n^2-1)k^2x^3\}\frac{dV}{dx}$$
$$+2n^2k(1-k^2)\frac{dV}{dk}+\{n^2(n^2-1)k^2x^2+n^2(1-k^2)\}V=0\,.$$

We may suppose the numerator of sn nu to be arranged according to the powers of k^2 and write

$$(136)\qquad xA(x^2)=\sum_{m=0}^{m=p}P_{2m}k^{2m}.$$

Then, denoting the operator $x\dfrac{d}{dx}$ by ϑ, we have

$$(137)\quad \frac{d^2P_{2m}}{dx}+(n^2(4m+1)-\vartheta^2)P_{2m}=((4m-3)n^2-2n^2\vartheta+\vartheta^2)P_{2m-2}$$
$$-x^2(n^2-\vartheta)(n^2-1-\vartheta)P_{2m-2}\,.$$

For $m=0$, we have

$$\frac{d^2P_0}{dx^2}+(n^2-\vartheta^2)\,P_0=0,$$

and, if we put

$$P_0=\sum_{r=0}\beta_r\,x^{2r+1},$$

then

$$\beta_r\,(2r+1)2r=-\beta_{r-1}(n^2-(2r-1)^2)\,.$$

And, since $\beta_0=n$, $\beta_r=(-1)^r\dfrac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-1)^2)}{(2r+1)\,!}$; thus we obtain the well-known series

$$(138)\qquad P_0=\sum_{r=0}(-1)^r\frac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-1)^2)}{(2r+1)\,!}\,x^{2r+1}\,.$$

For $m=1$, we have

$$\frac{d^2P_2}{dx^2}+(5n^2-\vartheta^2)P_2=(n^2-2n\vartheta+\vartheta^2)P_0+x^2(n^2-\vartheta)(n^2-1-\vartheta)P_0\,.$$

Since the lowest power of x in P_2 is 3, we write

$$P_2 = \sum_{r=1} \gamma_r \, x^{2r+1}\,, \tag{139}$$

then

$$\begin{aligned}2r(2r+1)\gamma_r &+ (5n^2-(2r^2-1))\,\gamma_{r-1}\\ &= (n^2-2n^2(2r-1)+(2r-1)^2)\beta_{r-1}-(n^2-2r+3)(n^2-2r+2)\beta_{r-2}\,.\end{aligned}$$

We find

$$\gamma_1 = -\frac{n(n^2-1)}{3\,!}, \quad \gamma_2 = \frac{n(n^2-1)}{5\,!}2(2n^2-3), \quad \gamma_3 = -\frac{n(n^2-1)(n^2-9)}{7\,!}3(3n^2-5),$$

....

Generally,

$$\begin{aligned}
&2r(2r+1)\gamma_r+(5n^2-(2r-1)^2)\gamma_{r-1}\\
&\quad = (-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}\left\{\begin{array}{l}(4r^2-10r+5)n^4\\ -(12r^2-30r+16)n^2\\ +(2r-3)(2r-1)\end{array}\right\}\\
&\quad = (-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}\left\{\begin{array}{l}-r(rn^2-(2r-1))\,(n^2-(2r-3)^2)\\ +(5n^2-(2r-1)^2)\,(r-1)\\ \qquad\qquad ((r-1)n^2-(2r-3))\end{array}\right\}\\
&\quad = (-1)^{r}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)\,(n^2-(2r-3)^2)}{(2r-1)\,!}r(rn^2-(2r-1))\\
&\qquad\quad +(5n^2-(2r-1)^2)(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}(r-1)\\
&\qquad\qquad\qquad ((r-1)n^2-(2r-3))\,,
\end{aligned}$$

or,

$$\begin{aligned}
&2r(2r+1)\left[\gamma_r-(-1)^r\frac{n(n^2-1)\ldots(n^2-(2r-3)^2)}{(2r+1)\,!}r(rn^2-(2r-1))\right]\\
&\quad = -(5n^2-(2r-1)^2)\left[\gamma_{r-1}-(-1)^{r-1}\frac{n(n^2-1)\ldots(n^2-(2r-5)^2)}{(2r-1)\,!}(r-1)\right.\\
&\qquad\qquad \left.((r-1)n^2-(2r-3))\right];
\end{aligned}$$

whence follows

$$(140)\qquad \gamma_r = (-1)^r \frac{n(n^2-1)(n^2-9)\ldots(n^2-(2r-3)^2)}{(2r+1)!} r(rn^2-(2r-1)).$$

For $m=2$, we have

$$\frac{d^2P_4}{dx^2}+(9n^2-\vartheta^2)P_4=(5n^2-2n^2\vartheta+\vartheta^2)P_2-x^2(n^2-\vartheta)(n^2-\vartheta-1)P_2.$$

Since the lowest power of x in P_4 is 5, we write

$$(141)\qquad P_4 = \sum_{r=2} \delta_r x^{2r+1};$$

then, $2r(2r+1)\delta_r+(9n^2-(2r-1)^2)\delta_{r-1}$

$$=((2r-1)^2-n^2(4r-7))\gamma_{r-1}-(n^2-2r+3)(n^2-2r+2)\gamma_{r-2},$$

whence we find

$$(142)\qquad \begin{aligned} \delta_2 &= \frac{n(n^2-1)(n^2-9)}{5!}, \\ \delta_3 &= -\frac{n(n^2-1)(n^2-9)}{7!}3(3n^2-5), \\ \delta_4 &= -\frac{n(n^2-1)}{9!}6(n^6+85n^4-671n^2+945), \\ \delta_5 &= \frac{n(n^2-1)(n^2-9)}{11!}2(247n^6+325n^4-13757n^2+23625). \end{aligned}$$

Thus the general law is not obvious as in the case of γ, and it seems to be impracticable to proceed further in this way.

Reverting to the formulae

$$A=\sum_{m=0}^{m=2p} A_{2m}x^{2m},\quad D=\sum_{m=0}^{m=2p} D_{2m}x^{2m},$$

since, by virtue of (109),

$$A_{2m}=(-1)^{\frac{n-1}{2}}D_{4p-2m}k^{-(2p-2m)},\quad D_{2m}=(-1)^{\frac{n-1}{2}}A_{4p-2m}k^{-(2p-2m)},$$

we need only determine either A or D. Let us take D and apply (134). We obtain $D_0=1$, $D_2=0$, $D_{4p}=(-1)^{\frac{n-1}{2}}nk^{2p}$, and, generally,

$$(143)\quad (2m+1)(2m+2)D_{2m+2}+4m\{(n^2-m)k^2-m\}D_{2m}+2n^2k(1-k^2)\frac{dD_{2m}}{dk} \\ +\{(n^2-2m+1)(n^2-2m+2)\}k^2D_{2m-2}=0\,.$$

When $m\leqq p$,

$$(144)\quad D_{2m}=D_{2m,0}(1+k^{2m})+D_{2m,2}(k^2+k^{2m-2})+\ldots+D_{2m,2r}(k^{2r}+k^{2m-2r})+\ldots\,,$$

the last term being $D_{2m,m}k^m$ or $D_{2m,m-1}(k^{m-1}+k^{m+1})$ according as m is even or odd. Substituting this in (143), we obtain

$$\begin{aligned}
&(2m+1)(2m+2)\{D_{2m+2,0}(1+k^{2m+2})+D_{2m+2,2}(k^2+k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m+2,2r}(k^{2r}+k^{2m-2r+})+\ldots\} \\
&+4m(n^2-m)\,\{D_{2m,0}(k^2+k^{2m+2})+D_{2m,2}(k^4+k^{2m})+\ldots \\
&\qquad\qquad +D_{2m,2r}(k^{2r+2}+k^{2m-2r+2})+\ldots\} \\
&-\quad 4m^2\quad \{D_{2m,0}(1+k^{2m})+D_{2m,2}(k^2+k^{2m-2})+\ldots+D_{2m,2r}(k^{2r}+k^{2m-2r})+\ldots\} \\
&+\quad 2n^2\quad \{D_{2m,0}(*+2mk^{2m})+D_{2m,2}(2k^2+(2m-2)k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m,2r}(2rk^{2r}+(2m-2r)k^{2m-2r})+\ldots\} \\
&-\quad 2n^2\quad \{D_{2m,0}(*+2mk^{2m+2})+D_{2m,2}(2k^4+(2m-2)k^{2m})+\ldots \\
&\qquad\qquad +D_{2m,2r}(2rk^{2r+2}+(2m-2r)k^{2m-2r+2})+\ldots\} \\
&+(n^2-2m+1)(n^2-2m+2)\{D_{2m-2,0}(k^2+k^{2m})+D_{2m-2,2}(k^4+k^{2m-2})+\ldots \\
&\qquad\qquad +D_{2m-2,2r}(k^{2r+2}+k^{2m-2r})+\ldots\} \\
&\qquad\qquad =0\,.
\end{aligned}$$

Hence

$$\begin{aligned}
(2m+1)(2m+2)D_{2m+2,0} &= 4m^2D_{2m,0}\,,\quad D_{2m,0}=0, \\
(2m+1)(2m+2)D_{2m+2,2} &= -4(n^2-m^2)D_{2m,2} \\
(2m+1)(2m+2)D_{2m+2,4} &= -4\{(m-1)n^2-m^2\}D_{2m,2}-4(2n^2-m^2)D_{2m,4}\,, \\
&\quad -(n^2-2m+1)(n^2-2m+2)D_{2m-2,2}\,, \\
(2m+1)(2m+2)D_{2m+2,6} &= -4\{(m-2)n^2-m^2\}D_{2m,4}-4(3n^2-m^2)D_{2m,6}\,, \\
&\quad -(n^2-2m+1)(n^2-2m+2)D_{2m-2,4}\,,
\end{aligned}$$

and, generally,

$$(145)\quad (2m+1)(2m+2)D_{2m+2,\,2r} = -4\{(m-r+1)n^2-m^2\}D_{2m,\,2r-2}$$
$$-4(rn^2-m^2)D_{2m,\,2r}-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,2r-2}\,,$$

the last coefficient being given by

$$(2m+1)(2m+2)D_{2m+2,\,m+1} = -8\left\{\frac{m+1}{2}n^2-m^2\right\}D_{2m,\,m-1}$$
$$-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,m-1}\,,$$

or

$$(2m+1)(2m+2)D_{2m+2,\,m} = -2\{(m+2)n^2-2m^2\}D_{2m,\,m-2}$$
$$-4\left(\frac{m}{2}n^2-m^2\right)D_{2m,\,m}-(n^2-2m+1)(n^2-2m+2)D_{2m-2,\,m-2}\,,$$

according as m is odd or even.

By means of (145), we find

(146)

$$D_0 = 1,\quad D_2 = 0,\quad D_4 = -\frac{n^2(n^2-1)}{4\,!}2k^2,$$

$$D_6 = \frac{n^2(n^2-1)(n^2-4)}{6\,!}8(k^2+k^4),$$

$$D_8 = -\frac{n^2(n^2-1)(n^2-4)}{8\,!}4\{8(n^2-9)(k^2+k^6)+(17n^2-69)k^4\},$$

$$D_{10} = \frac{n^2(n^2-1)(n^2-4)(n^2-9)}{10\,!}32\{4(n^2-16)(k^2+k^8)+15(n^2-4)(k^4+k^6)\},$$

$$D_{12} = -\frac{n^2(n^2-1)(n^2-4)(n^2-9)}{12\,!}8\{64(n^2-16)(n^2-25)(k^2+k^{10})$$
$$+8(n^2-16)(47n^2-185)(k^4+k^8)+15(45n^4-569n^2+1544)k^6\},$$

$$D_{14} = \frac{n^2(n^2-1)(n^2-4)(n^2-9)}{14\,!}32\{64(n^2-16)(n^2-25)(n^2-36)(k^2+k^{12})$$
$$+16(n^2-16)(34n^4-982n^2+3300)(k^4+k^{10})$$
$$+(1549n^6-43925n^4+357196n^2-815040)(k^6+k_8)\}.$$

Again equation (143) may be written in the form

$$(147)\quad (2m+2)(2m+3)k^2D_{4p-2m-2}+2n^2k(1-k^2)\frac{dD_{4p-2m}}{dk}$$
$$+(n^2-2m-1)\{(n^2+2m+1)k^2-(n^2-2m-1)\}D_{4p-2m}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2}=0\,.$$

Now, m being less than p,

$$(148)\quad D_{4p-2m}=D_{4p-2m,\,2p-2m}(k^{2p-2m}+k^{2p})+D_{4p-2m,\,2p-2m+2}(k^{2p-2m+2}+k^{2p-2})+\ldots$$
$$+D_{4p-2m,\,2p-2m+2r}(k^{2p-2m+2r}+k^{2p-2r})+\ldots\,,$$

the last term being

$$D_{4p-2m,\,2p-m}\,k^{2p-m}\quad\text{or}\quad D_{4p-2m,\,2p-m-1}(k^{2p-m-1}+k^{2p-m+1})\,,$$

according as m is even or odd.

From (147) and (148), we obtain

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m-2}+\{n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m}=0\,,$$

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m}+\{5n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2}$$
$$+\{(4m+1)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+2}=0\,,$$

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-2m+2}+\{9n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+4}$$
$$+\{(4m-3)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+4}=0\,,$$

generally,

$$(149)\quad (2m+2)(2m+3)D_{4p-2m-2,\,2p-2m+2r-2}$$
$$+\{(4r+1)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2r}$$
$$+\{(4m-4r+5)n^2-(2m+1)^2\}D_{4p-2m,\,2p-2m+2r-2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-2m+2r}=0\,,$$

and, lastly,

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-m-2}+(2m+1)(n^2-2m-1)D_{4p-2m,\,2p-m}$$
$$+\{(2m+5)n^2-(2m+1)^2\}D_{4p-2m,\,2p-m-2}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-m}=0\,,$$

or

$$(2m+2)(2m+3)D_{4p-2m-2,\,2p-m-1}+2\{(2m+3)n^2-(2m+1)^2\}D_{4p-2m,\,2p-m-1}$$
$$+(n^2-2m)(n^2-2m+1)D_{4p-2m+2,\,2p-m+1}=0\,,$$

according as m is even or odd.

From the above equations, we find

(150)

$$D_{4p-12}=(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)(n^2-9)}{13\,!}k^{2p-12}\{(n^2-25)(n^2-49)(n^2-81)(n^2-121)(1+k^{12})$$
$$+6(n^2-25)(n^2-49)(n^2-81)(6n^2-11)(k^2+k^{10})$$
$$-3(1927n^8-33068n^6-962n^4$$
$$+1033308n^2-1819125)(k^4+k^8)$$
$$-4(3046n^8-38037n^6+32799n^4$$
$$+708647n^2-1299375)k^6\}\,,$$

$$D_{4p-10}=-(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)(n^2-9)}{11\,!}k^{2p-10}\{(n^2-25)(n^2-49)(n^2-81)(1+k^{10})$$
$$+5(n^2-25)(n^2-49)(5n^2-9)(k^2+k^8)$$
$$-2(247n^6+325n^4-13757n^2+23625)(k^4+k^6)\}\,,$$

$$D_{4p-8}=(-1)^{\frac{n-1}{2}}\frac{n(n^2-1)}{9\,!}k^{2p-8}\{(n^2-9)(n^2-25)(n^2-49)(1+k^8)$$
$$+4(n^2-9)(n^2-25)(4n^2-7)(k^2+k^6)$$
$$-6(n^6+85n^4-671n^2+945)k^4\}\,,$$

$$D_{4p-6} = -(-1)^{\frac{n-1}{2}} \frac{n(n^2-1)(n^2-9)}{7\,!} k^{2p-6}\{(n^2-25)(1+k^6)+3(3n^2-5)(k^2+k^4)\},$$

$$D_{4p-4} = \quad (-1)^{\frac{n-1}{2}} \frac{n(n^2-1)}{5\,!} k^{2p-4}\{(n^2-9)(1+k^4)+2(2n^2-3)k^2\},$$

$$D_{4p-2} = -(-1)^{\frac{n-1}{2}} \frac{n(n^2-1)}{3\,!} k^{2p-2}(1+k^2)\,,$$

$$D_{4p} \quad = \quad (-1)^{\frac{n-1}{2}} n\, k^{2p}.$$

The first five coefficients were given by Jacobi himself. The first and the last six coefficients of all the four functions have been found by Baehr by a different method.

§ 17.

Let us now consider the rational integral functions of x^2, B, C which enter into the numerators of $\operatorname{cn} nu$ and $\operatorname{dn} nu$, n being odd.

From the well-known relations

$$(151) \qquad B\,(x,k) = C\left(\frac{1}{kx}, k\right) k^{2p}\, x^{4p}, \quad C\,(x,k) = B\left(\frac{1}{kx}, k\right) k^{2p}\, x^{4p},$$

$$(152) \qquad B\left(kx, \frac{1}{k}\right) = C\,(x,k), \qquad\qquad C\left(kx, \frac{1}{k}\right) = B\,(x,k),$$

we deduce

$$(153) \qquad B\left(kx, \frac{1}{k}\right) = B\left(\frac{1}{kx}, k\right) k^{2p}\, x^{4p},$$

whence

$$(154) \qquad B_{4p-2m}(k) = B_{2m}\left(\frac{1}{k}\right) k^{2p}, \quad B_{2m}(k) = B_{4p-2m}\left(\frac{1}{k}\right) k^{2p}.$$

Hence, if we put

$$B = \sum_{m=0}^{m=2p} B_{2m}(k)\, x^{2m},$$

then

$$C = \sum_{m=0}^{m=2p} B_{2m}\left(\frac{1}{k}\right) x^{2m},$$

so that we need only determine B_{2m} and this only for the initial values of m in view of **(154)**.

Now **(134)** may easily be modified in such a manner that the resulting equation is satisfied by B. We find

$$\text{(155)}\quad \{1-(1+k^2)x^2+k^2x^4\}\frac{d^2B}{dx^2}+\{[(2n^2-1)k^2-3]x-2(n^2-2)k^2x^3\}\frac{dB}{dx}$$
$$+2n^2k(1-k^2)\frac{dB}{dk}+(n^2-1)\{1+(n^2-2)k^2x^2\}B=0,$$

whence follows

$$\text{(156)}\quad (2m+1)(2m+2)B_{2m+2}+\{[n^2-(2m+1)^2]+4m(n^2-m)k^2\}B_{2m}$$
$$+2n^2k(1-k^2)\frac{dB_{2m}}{dk}+(n^2-2m)(n^2-2m+1)k^2B_{2m-2}=0.$$

Further we find, for $m \leqq p$,

$$(2m+1)(2m+2)B_{2m+2,0}+\{n^2-(2m+1)^2\}B_{2m,0}=0,$$
$$B_{2m,0}=0, \qquad n-1<2m\leqq 2p,$$
$$(2m+1)(2m+2)B_{2m+2,2}+\{5n^2-(2m+1)^2\}B_{2m,2}$$
$$+4m(n^2-m)B_{2m,0}+(n^2-2m)(n^2-2m+1)B_{2m-2,0}=0,$$
$$(2m+1)(2m+2)B_{2m+2,4}+\{9n^2-(2m+1)^2\}B_{2m,4}$$
$$+4\{(m-1)n^2-m^2\}B_{2m,2}+(n^2-2m)(n^2-2m+1)B_{2m-2,2}=0,$$

generally,

$$(175)\quad (2m+1)(2m+2)B_{2m+2,\,2r}+\{(4r+1)n^2-(2m+1)^2\}B_{2m,\,2r}$$
$$+4\{(m-r+1)n^2-m^2\}B_{2m,\,2r-2}$$
$$+(n^2-2m)(n^2-2m+1)B_{2m-2,\,2r-2}=0\,,$$

and, lastly,

$$\{(4m+1)n^2-(2m+1)^2\}B_{2m,\,2m}+(n^2-2m)(n^2-2m+1)B_{2m-2,\,2m-2}=0\,,$$

$$B_{2m,\,2m}=0\,,\qquad m\leqq p\,.$$

By means of the above equations, we find

(158)

$$B_0=1,\quad B_2=-\frac{n^2-1}{2\,!},$$

$$B_4=\frac{n^2-1}{4\,!}\{(n^2-9)+2n^2k^2\},$$

$$B_6=-\frac{n^2-1}{6\,!}\{(n^2-9)(n^2-25)+6n^2(n^2-9)k^2+8n^2(n^2-4)k^4\},$$

$$B_8=\frac{n^2-1}{8\,!}\{(n^2-9)(n^2-25)(n^2-49)+12n^2(n^2-9)(n^2-25)k^2$$
$$+4n^2(n^2-4)(15n^2-107)k^4+32n^2(n^2-4)(n^2-9)k^6\},$$

$$B_{10}=-\frac{(n^2-1)(n^2-9)}{10\,!}\{n^2-25)(n^2-49)(n^2-81)+20n^2(n^2-25)(n^2-49)k^2$$
$$+12n^2(n^2-4)(29n^2-329)k^4+32n^2(n^2-4)(14n^2-89)k^6$$
$$+128n^2(n^2-4)(n^2-16)k^8\},$$

$$B_{12}=\frac{(n^2-1)(n^2-9)}{12\,!}\{(n^2-25)(n^2-49)(n^2-81)(n^2-121)$$
$$+30n^2(n^2-25)(n^2-49)(n^2-81)k^2$$
$$+4n^2(593n^6-17082n^4+179517n^2-482708)k^4$$
$$+8n^2(n^2-4)(575n^4-10111n^2+44276)k^6$$
$$+192n^2(n^2-4)(n^2-16)(15n^2-89)k^8$$
$$+512n^2(n^2-4)(n^2-16)(n^2-25)k^{10}\}.$$

§ 18.

Consider next the case where n is even. To begin with, take $\operatorname{sn} nu$. Here D is exactly of the same form as in the case where n is odd, so that we may restrict ourselves to the consideration of A alone.

Now A is of the degree $n^2-4=4p$ say, and

(159) $$A(x, k) = (-1)^{\frac{n-2}{2}} A\left(\frac{1}{kx}, k\right) k^{2p} x^{4p},$$

and, therefore,

(160) $$A_{4p-2m} = (-1)^{\frac{n-2}{2}} A_{2m}\, k^{4p-2m}.$$

In consequence of (104) and (134), $A = \sum\limits_{m=0}^{m=2p} A_{2m} x^{2m}$ satisfies the differential equation

(161) $$\{x-(1+k^2)x^3+k^2x^5\}\frac{d^2A}{dx^2}+\{2+[(2n^2-5)k^2-5]x^2-2(n^2-4)k^2x^4\}\frac{dA}{dx}$$
$$+2n^2k(1-k^2)x\frac{dA}{dk}+(n^2-4)\{(1+k^2)x+(n^2-3)k^2x^3\}A=0\,,$$

whence follows,

(162) $$(2m+2)(2m+3)A_{2m+2}+\{n^2-4(m+1)^2+[(4m+1)n^2-4(m+1)^2]k^2\}A_{2m}$$
$$+2n^2k(1-k^2)\frac{dA_{2m}}{dk}+(n^2-2m-2)(n^2-2m-1)k^2A_{2m-2}=0\,.$$

By virtue of (160), we need only determine the first half of the coefficients A_{2m}. Again, A_{2m} is of the form

$$A_{2m} = A_{2m,0}(1+k^{2m})+A_{2m,2}(k^2+k^{2m-2})+\ldots+A_{2m,2r}(k^{2r}+k^{2m-2r})+\ldots\,,$$

the last term being $A_{2m,m}k^m$ or $A_{2m,m-1}(k^{m-1}+k^{m+1})$, according as m is even or odd. Substituting this in (162), we get

$$(2m+2)(2m+3)A_{2m+2,0}+\{n^2-4(m+1)^2\}A_{2m,0}=0\,,$$

$$A_{2m,0}=(-1)^m\frac{n(n^2-4)(n^2-16)\ldots(n^2-4m^2)}{(2m+1)\,!}\,,$$

$$(2m+2)(2m+3)A_{2m+2,2}+\{5n^2-4(m+1)^2\}A_{2m,2}$$
$$+\{(4m+1)n^2-4(m+1)^2\}A_{2m,0}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,0}=0\,,$$

$$(2m+2)(2m+3)A_{2m+2,4}+\{9n^2-4(m+1)^2\}A_{2m,4}$$
$$+\{(4m-3)n^2-4(m+1)^2\}A_{2m,2}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,2}=0\,,$$

$$(2m+2)(2m+3)A_{2m+2,6}+\{13n^2-4(m+1)^2\}A_{2m,6}$$
$$+\{(4m-7)n^2-4(m+1)^2\}A_{2m,4}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,4}=0\,,$$

generally,

(163) $$(2m+2)(2m+3)A_{2m+2,2r}+\{(4r+1)n^2-4(m+1)^2\}A_{2m,2r}$$
$$+\{(4m-4r+5)n^2-4(m+1)^2\}A_{2m,2r-2}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,2r-2}=0\,,$$

and, lastly,

$$(2m+2)(2m+3)A_{2m+2,m+1}+2\{(2m+3)n^2-4(m+1)^2\}A_{2m,m-1}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,m-1}=0\,,$$

or

$$(2m+2)(2m+3)A_{2m+2,m}+\{(2m+1)n^2-4(m+1)^2\}A_{2m,m}$$
$$+\{(2m+5)n^2-4(m+1)^2\}A_{2m,m-2}$$
$$+(n^2-2m-2)(n^2-2m-1)A_{2m-2,m-2}=0\,,$$

according as m is odd or even.

By means of the above equations, we find

$$(164)$$

$$A_0 = n, \quad A_2 = -\frac{n(n^2-4)}{3!}(1+k^2),$$

$$A_4 = \frac{n(n^2-4)}{5!}\{(n^2-16)(1+k^4)+2(2n^2-7)k^2\},$$

$$A_6 = -\frac{n(n^2-4)(n^2-16)}{7!}\{(n^2-36)(1+k^6)+3(3n^2-10)(k^2+k^4)\},$$

$$A_8 = \frac{n(n^2-4)}{9!}\{(n^2-16)(n^2-36)[(n^2-64)(1+k^8)+4(4n^2-13)(k^2+k^6)] \\ -6(n^6+196n^4-2114n^2+4752)k^4\},$$

$$A_{10} = -\frac{n(n^2-4)}{11!}\{(n^2-16)(n^2-36)(n^2-64)[(n^2-100)(1+k^{10})+5(5n^2-16)(k^2+k^8)] \\ -2(247n^8-882n^6-72102n^4+661112n^2-1368000)(k^4+k^6)\},$$

$$A_{12} = \frac{n(n^2-4)}{13!}\{(n^2-16)(n^2-36)(n^2-64)(n^2-100)(n^2-144)(1+k^{12}) \\ +6(n^2-16)(n^2-36)(n^2-64)(n^2-100)(6n^2-19)(k^2+k^{10}) \\ +(-5781n^{10}+170472n^8-490140n^6-22744752n^4 \\ +193986816n^2-383754240)(k^4+k^8) \\ +4(-3046n^{10}+75579n^8-260532n^6-5901554n^4 \\ +48907368n^2-93493440)k^6\}.$$

§ 19.

Lastly we consider the numerators of $\operatorname{cn} nu$ and $\operatorname{dn} nu$, n being even. In this case, put $n^2 = 4p$.

As in the case where n is odd, $C_{2m}(k) = B_{2m}\left(\frac{1}{k}\right)k^{2m}$ by (106), so

that we may here also restrict ourselves to the consideration of B. Moreover, since in this case

$$B(x, k) = B\left(\frac{1}{kx}, k\right) k^{2p} x^{4p}, \tag{165}$$

it follows,

$$B_{4p-2m} = B_{2m} k^{2p-2m}, \tag{166}$$

and we need only determine the first half of the coefficients B_{2m}.

Now B satisfies the differential equation

$$\{1-(1+k^2)x^2+k^2x^4\}\frac{d^2B}{dx^2}+\{[(2n^2-1)k^2-1]x-2(n^2-1)k^2x^3\}\frac{dB}{dx} \tag{167}$$
$$+2n^2k(1-k^2)\frac{dB}{dk}+\{n^2(n^2-1)k^2x^2+n^2\}B = 0,$$

whence we deduce

$$(2m+1)(2m+2)B_{2m+2}+\{(n^2-4m^2)+4m(n^2-m)k^2\}B_{2m} \tag{168}$$
$$+2n^2k(1-k^2)\frac{dB_{2m}}{dk}+(n^2-2m+1)(n^2-2m+2)k^2B_{2m-2} = 0.$$

Substituting

$$B_{2m} = B_{2m,0} + B_{2m,2}k^2 + \ldots + B_{2m,2r}k^{2r} + \ldots + B_{2m,2m}k^{2m},$$
$$m \leqq p,$$

in (168), we get

$$(2m+1)(2m+2)B_{2m+2,0} + (n^2-4m^2)B_{2m,0} = 0,$$
$$B_{2m,0} = 0, \qquad n < 2m \leqq 2p,$$

$$(2m+1)(2m+2)B_{2m+2,2} + (5n^2-4m^2)B_{2m,2}$$
$$+4m(n^2-m)B_{2m,0} + (n^2-2m+1)(n^2-2m+2)B_{2m-2,0} = 0,$$

$$(2m+1)(2m+2)B_{2m+2,\,4}+(9n^2-4m^2)B_{2m,\,4}$$
$$+4\{(m-1)n^2-m^2\}B_{2m,\,2}+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2}=0\,,$$

generally,

(169) $$(2m+1)(2m+2)B_{2m+2,\,2r}+\{(4r+1)n^2-4m^2\}B_{2m,\,2r}$$
$$+4\{(m-r+1)n^2-m^2\}B_{2m,\,2r-2}$$
$$+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2r-2}=0\,,$$

and, lastly,

$$(2m+1)(2m+2)B_{2m+2,\,2m+2}+\{(4m+1)n^2-4m^2\}B_{2m,\,2m}$$
$$+(n^2-2m+1)(n^2-2m+2)B_{2m-2,\,2m-2}=0\,,$$

By means of the above equations, we find

(170)

$$B_0=1,\quad B_2=-\frac{n^2}{2\,!},$$

$$B_4=\frac{n^2}{4!}\{(n^2-4)+2(n^2-1)k^2\},$$

$$B_6=-\frac{n^2(n^2-4)}{6\,!}\{(n^2-16)+6(n^2-1)k^2+8(n^2-1)k^4\},$$

$$B_8=\frac{n^2(n^2-4)}{8\,!}\{(n^2-16)(n^2-36)+12(n^2-1)(n^2-16)k^2+$$
$$+4(n^2-1)(15n^2-51)k^4+32(n^2-1)(n^2-9)k^6\},$$

$$B_{10}=-\frac{n^2(n^2-4)}{10\,!}\{(n^2-16)(n^2-36)(n^2-64)+20(n^2-1)(n^2-16)(n^2-36)k^2$$
$$+12(n^2-1)(n^2-9)(29n^2-104)k^4$$
$$+64(n^2-1)(n^2-9)(7n^2-22)k^6$$
$$+128(n^2-1)(n^2-9)(n^2-16)k^8\},$$

$$B_{12} = \frac{n^2(n^2-4)}{12!}\{(n^2-16)(n^2-36)(n^2-64)(n^2-100)$$
$$+30(n^2-1)(n^2-16)(n^2-36)(n^2-64)k^2$$
$$+4(n^2-1)(593n^6-14305n^4+113972n^2-257760)k^4$$
$$+40(n^2-1)(n^2-9)(115n^4-1283n^2+2968)k^6$$
$$+2880(n^2-1)(n^2-9)(n^2-16)(n^2-3)k^8$$
$$+512(n^2-1)(n^2-9)(n^2-16)(n^2-25)k^{10}\}.$$

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On the Process of Gastrulation in Chelonia.

(Contributions to the Embryology of Reptilia, IV.)

By

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With Plates VI.—VIII.

The sea-turtle, Chelonia caouana, *Wagl.* deposits its eggs on almost every suitable stretch of sandy beach in the southern half of Japan during the summer months of the year. During the breeding season of this animal in 1891, I was enabled, by the liberality of the University authorities, to visit Sagara in the province of Tōtōmi, with my assistant, Mr. T. Tsuchida, for the purpose of collecting materials for the study of its development. With the assistance of several kind friends, we made arrangements to have reported to us every deposit of eggs that might be made along about fifteen miles of sandy beach in that region, and we thus succeeded in getting hold of several good deposits. As each of these contained over one hundred eggs—121 is the least, and 145, the largest number in one deposit in my experience, but 169 has been reported in one case—we had command of over one thousand eggs in all, and as we opened eggs from each deposit at certain intervals of time, we were able to secure unusually fine series of embryos, gaps in one series being often filled up by members

from others. This success was in a large measure due to Mr. Tsuchida, whose zeal and perseverance never flagged, even under most trying circumstances; and I would here express my deep indebtedness to him. My thanks are also due to Viscount Tanuma, Mr. Y. Murakami, the Mayor of Sagara, the Chief Officer of the Sagara Police Station, and several other gentlemen who assisted us in various ways and showed us much kindness during our stay. Messrs. T. Ogasawara and K. Niwa of Shizuoka were also kind enough to furnish me with much useful information.

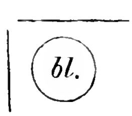
Various observation made by us on the breeding habits of the sea-turtle together with similar facts which I have ascertained in other species, I hope to embody, at some future time, in a separate paper. A short preliminary account of these observations is already published in the Zoölogical Magazine (Japanese) Vol. III., No. 35. I will only remark here for the benefit of those who may attempt a similar study, that Chelonian eggs can be transported with safety for some hours immediately following their deposition, but after that, their removal is apt to bring on death and decomposition. This seems to be due to the circumstance that the white at the upper pole is rapidly absorbed, the blastoderm becomes adherent to the shell membrane, and a large fluid cavity is produced directly beneath the developing embryo. In this condition, slight jarring seems to disturb the delicate arrangements and to cause death. After thirty days or so, when the fœtal membranes have become definitely established, the eggs can again be moved with impunity.

The embryos of Chelonia caouana, thus obtained, together with those of Trionyx japonicus and Clemmys japonica which I already possess or can get in almost any desired stage, afford a good basis for the comparative study of reptilian development, and I intend to use them for this purpose, as I have previously used those

of the two last-named species. Meanwhile I have discovered that when Chelonia caouana deposits its eggs, they are in a far less advanced condition than those of Trionyx or Clemmys and thus enable us to elucidate many points in the much discussed process of gastrulation in the Amniota. The present contribution embodies the results of my own study on this point and, it is hoped, will throw light on some phases of this vexed question.

Preparation and Preservation of the Embryos.

Young embryos were in nearly all cases preserved in Kleinenberg's picro-sulphuric acid. Very advanced embryos were placed, partly in that fluid, and partly in corrosive sublimate. In removing blastoderms from eggs within one or two days of their deposition, at which age there is not yet any large subgerminal cavity in the yolk, the shell was removed and as much of the white as possible. The whole egg was then placed with the blastoderm uppermost in a deep vessel and covered with picro-sulphuric acid. The spot where the blastoderm was to be found was generally marked with a hair since the thin layer of the white necessarily left over it coagulates in the preserving fluid and hides it entirely from view. Proceeding in this way the preserving fluid will be found after three or four hours to have penetrated to the blastoderm and acted on it as also on the upper strata of the yolk. Incisions at right angles were now made with a sharp knife on three sides of the blastoderm, leaving the fourth side and the two corners uncut, as shown in the accompanying diagram. It was then found that a little manipulation with forceps or scalpel easily separates the superficial coagulated white from the blastoderm beneath it. If we then cut the corners, the sheet of the white will roll up of itself towards the uncut side, leaving fully exposed the blastoderm which being already hardened



can then be removed with great ease. The blastoderm thus removed was generally left in a relatively large quantity of the preserving fluid for some hours longer. In more advanced embryos the position of the blastoderm under the shell is easily told in chelonian eggs by the change of colour in the shell. In all the species I have examined, a white patch appears in the shell over the embryo, and increases in extent with the growth of the embryo, or more strictly speaking, *pari passu* with the disappearance of the white over the embryo ; so that, roughly speaking, the size of the patch is a very good indication of the size of the embryo beneath. In these stages the embryo is firmly adherent to the inside of the shell, with a large subgerminal fluid cavity in the yolk beneath it, which can be easily pierced through the shell and the blastoderm with the point of one blade of the fine scissors. By thus piercing the cavity and cutting round just inside the edge of the white patch through both shell and blastoderm, the embryo is removed, firmly adhering to the cut piece ; the latter can then be turned over, exposing the ventral surface of the embryo, and the preserving fluid be poured over it, using the cut-piece as a veritable watch-glass. After half an hour or so, the blastoderm can be easily separated from the shell and placed in a larger quantity of the preserving fluid. This method has the great merit of keeping every part of the blastoderm stretched in its natural condition, and also of making it possible to remove a large number of embryos in an incredibly short space of time.

When the embryo is very much advanced and the allantois has spread itself entirely beneath the shell, it becomes a serious question how to remove the shell without much injury to the fœtal membranes, especially as the shell is leathery, and not brittle as in some other reptiles. In this and similar cases I carefully scrape the shell at one small spot with a knife, until it becomes quite thin, and then apply

to that spot some picro-sulphuric acid, which removes calcareous matter. I scrape again with the knife and again apply the acid. I repeat this process, always using great care, until enough of the shell is worn off to expose a very small patch of the allantoic surface, sometimes not larger than the eye of a needle. However small the opening may be, the acid is able to penetrate through it and harden the tissues for some space around it. The opening may then be enlarged a little, with perfect safety to the parts beneath. The acid is then applied again, a still large area is hardened, and the opening is accordingly made still larger. At length the opening becomes large enough to allow of the removal of the entire shell without injury to the membranes. In removing the shell, it is advisable to use the broad, blunt-pointed forceps and insert them tangentially between the shell and the fœtal membranes. With a little practice, it becomes comparatively easy to obtain in this manner embryos with the fœtal membranes perfect, except for the yellow patch where the picro-sulphuric acid was first applied.

As to staining, imbedding, and cutting sections, there is nothing special to communicate. I generally use borax-carmine for staining. For imbedding, celloidin-paraffin is used.

The methods just described have been used in the case of Chelonia caouana and in those of other species with equal success.

Description of the stages of Gastrulation in Chelonia Caouana.

The first stage to which I wish to call attention is represented in Figs. 1 and 1*a*, Pl. VI. It was taken out of an egg which had been deposited only a few hours before. We notice first the oval-shaped embryonic shield somewhat elongated in the antero-posterior direction. At the posterior end of this, and for the most part, lying outside it,

there is a second much smaller, irregularly circular white patch. This is the structure called the "Primitivplatte" (the primitive plate) by Will* (No. 18); and in later stages, when cells added on from the subjacent yolk form an accumulation, is the "Primitivknoten" (the primitive knob) of Mehnert (No. 8). These names will be adopted and used interchangeably in this paper, for it is after all difficult to distinguish when the state of the plate ends and that of the knob begins. One stage of it is also called the "Sichel" by Will (No. 18). In the dorsal view (Fig. 1) there is already in the middle of this area a large, transversely elongated opening leading into a spacious cavity. For convenience in description, I anticipate my conclusions by stating here that I consider this cavity to be the archenteron and its dorsal opening to be the blastopore. In the ventral view (Fig. 1*a*), the appropriateness of calling the above area a "knob" is clearly seen, for it is a thick accummulation of cells projecting much more than the adjacent parts into the yolk. It is important above all to notice that the archenteron at this stage is not open on the ventral side, although it can be seen from that side through the cell-mass in the specimen figured.

Fig. 9, Pl. VIII. is a median longitudinal section through another embryo of the same stage as that represented in Figs. 1 and 1*a*. In this the yolk was left intact on the ventral face, so that it represents no doubt a more nearly normal state of things than if the yolk had been removed as in Fig. 1*a*. The blastoderm has already extended itself over a wide area and, with the exception of the primitive plate, is divided throughout into two layers: (*a*) the superficial layer (the epiblast of authors, the blastophor of Van Beneden), and

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* Also in No. 21. I regret that this paper came to my hands only after the present contribution was nearly in shape to be given to the printer. I could not therefore make as much use of it as I should have liked to do.

(*b*) the lower layer (the cœnogenetic hypoblast of Hubrecht, the paraderm of Kupffer, and of Mehnert, the lecitophor of Van Beneden.) The superficial layer, which I shall call the epiblast, forms a distinct membrane and is composed of columnar cells in the region of the embryonic shield, but changes gradually into low cells in the parts outside the shield. The lower layer is composed of irregular, amœboid-shaped cells and does not probably form a continuous membrane. I hope to show in the sequel that this layer ought to be regarded as only a part of the hypoblast, and might be called the cœnogenetic hypoblast, after Hubrecht (No. 5). In the region of the primitive plate, there is a different state of things. Instead of having two distinct layers, this area shows a thick accumulation of cells. It is composed for the most part of an irregular network of cells with tolerably wide meshes between, so that it is not a compact mass. In the middle of this accumulation, there is seen the invagination-cavity—the archenteron—leading at first downwards but soon forwards and ending blindly. The roof of this cavity shows distinctly a columnar arrangement of cells, and becomes continuous with the epiblast at the anterior lip of the blastopore. On the floor, as well as for some distance behind the blastopore, so long as we are in the region of the primitive plate, we see no columnar arrangement: the general network of the mass extends up to the surface. There is no sharp line of demarcation between the cellular mass of the primitive plate and the subjacent bed of the yolk. The latter is divided into especially fine globules at the boundary line, and we can clearly see many cells arising at this place and adding themselves to the primitive knob. That the nuclei of these cells are the descendants of the segmentation nucleus, there can be no reasonable doubt; in fact I would have this addition of new cells considered simply as the continuation of the process

of segmentation.* As the figure shows, the primitive plate becomes continuous with both the epiblast and the lower layer at its own periphery. I am thus unable to find any independent sheet of cells which lies below the primitive plate, and with which alone the lower layer of the surrounding parts becomes continuous, as Wenckebach (No. 15, Fig. 1) and Mehnert (No. 8, Figs. 20 and 21) and Will (No. 21, Fig. 49) have found. On this point Virchow's observations (No. 14) seem to be similar to mine; Will also sees no such independent sheet of cells in earlier stages (No. 19 Fig. 1, No. 21). I am also unable to find such a sharp line of demarcation between the shield and the primitive plate as is given in Will's Fig. 1 (No. 19†): the primitive plate passes gradually into the epiblast both anteriorly and posteriorly.

The yolk globules in this series of sections are generally spheroidal with a uniform yellow tint. Generally speaking, they are markedly fine immediately below the blastoderm, and become larger farther below (See Fig. 9): of their especially small size underneath the primitive knob, I have already spoken. I agree with Virchow (No. 14, p. 67) in thinking that the dark granules which Mehnert describes in the yolk-globules (No. 8, Figs. 20 and 21) are artificial productions made in the course of preparing sections.

I regret that I was unable to obtain the growing edge of the blastoderm which would no doubt present interesting phases of growth as observed by Virchow (No. 14) and Duval (No. 2). I am therefore unable to throw much light on many questions bearing on cells found in the yolk. I may however mention that in the series from which Fig. 9 is taken, two kinds of cells are found imbedded in the yolk. Their nature is not clear to me. The more numerous kind I

* Will appears to be of the same opinion (No. 21).

† This sharp demarcation is insisted on still more strongly in Will's recent paper (No. 21).

have represented in Figs. 10 and 12. Fig. 10 is part of a section like Fig. 9 from a more lateral part of the blastoderm than that in the latter figure. Beginning from the upper surface, we can easily recognise the epiblast and the lower layer of cells lying immediately below. Under these two layers, there is a rather thick stratum of spherical yolk-globules. We then come to a crowd of cells which are the cells in question. Some of these are large and full of yolk-granules ; while others are smaller and formed of vacuolated protoplasm. The size of their nuclei is tolerably uniform—being about .016 mm. in length. In Fig. 12*b* are shown more distinctly three of these cells from another region. One of them is full of yolk-granules, as is also the unusually large cell shown in Fig. 12*a*. Another is partly full, with an area of granular protoplasm around the nucleus. The third, of which only one-half is seen—having no doubt been cut in two in the process of microtomizing—has no yolk-globules, but is formed of vacuolated protoplasm. Below this stratum of cells there is a layer of closely packed fine granules which represents some liquid coagulated in the course of hardening. Below this, we come to the thick bed of yolk. The globules are here larger than in the upper layer. The conclusion seems to me almost inevitable that the cells above described take up and digest yolk globules and that the stratum of liquid on the edge of which they are found is produced as the result of their digestion. This liquid stratum has probably a genetic relation with the large subgerminal liquid cavity found below the blastoderm a day or two later in the course of development. So much seems tolerably clear ; but whether the cells have for their sole end the digestion and preparation of yolk globules for the nutrition of other cells, or whether they themselves are to form some integral parts of the growing embryo, I am unable to decide.

Cells of the second kind found in the yolk never occur together

in large numbers but are scattered, at least near the surface, indifferently through the yolk-substance. At different points in the yolk we find unusually large nuclei surrounded by a comparatively small amount of protoplasm (Fig. 11*a* and *b*). Sometimes there is only one nucleus (Fig. 11*b*) and then it is very large. The one represented in Fig. 11*b* measures .04 × .032 mm. Quite as often, the nuclei occur in a group of two or three, closely adherent to one another (Fig. 11*a*). These cells are no doubt what are called "Merocyten" by Virchow (No. 14). What their nature is, whether they stand in some genetic relation to other kinds of cells or are of a nature *sui generis*, I am unable to say. I have thought it just possible from the frequency with which two or three nuclei are found together, that they are cells dividing by amitosis and possibly undergoing disintegration (Flemming [No. 4] and Ziegler [No. 20]).

Let us consider for a moment how such a stage as that described has been reached. What I am inclined to think as probable is as follows :—When the process of segmentation has gone on for some time, the blastoderm separates itself into two layers, the superficial epiblast and the lower layer. This takes place throughout the blastoderm with the exception of the primitive plate.* Here cells not only remain undifferentiated but with the addition of cells from the subjacent bed of yolk form a mass which protrudes into the yolk—the primitive knob. In the middle of this region, an invagination soon appears, which is at first shallow and is directed straight downwards. I have two specimens of this stage but have not figured them because the blastoderms having been peeled off from the yolk to which it is adherent at this stage, the lower part of it is

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* And probably also of the growing edge of the blastoderm, but of this part I am not now, speaking——I am gratified to find that what is given above as probable is now verified by Will by direct observation (See No. 21, Figs. 35 & 36).

probably not complete. But as to the above point there is not room for much doubt. The specimens are very much like Fig. 1 of Will (No. 19) with one exception, stated above, viz: that the epiblast of the shield is continuous with the primitive plate and not separate as in Will's figure. One peculiarity of this stage is that both the anterior and the posterior wall of the invagination shows faintly the columnar arrangement as seen in Will's figure. Later on, this feature is confined to the anterior or dorsal wall (Figs. 9, 13, 14). After going straight downwards some distance the invagination cavity takes a forward horizontal direction and reaches the condition shown in Fig. 9. At the anterior lip of the blastopore, the columnar cells are recognizable very early, and the epiblast is here reflected downwards to become continuous with the anterior or dorsal wall of the invagination. In that part of the primitive plate placed behind the invagination the cell-mass remains undifferentiated for a long time, there being later established in this place the rudimentary yolk-plug, as was minutely described in the joint paper of Ishikawa and myself* on the germinal layers of Trionyx. Robinson and Assheton (No. 10) object to our idea of considering the structure in question as the yolk-plug. In the course of this paper, I hope to show that the presence of the yolk-plug at this place is an important feature in homologising the gastrulation of the Sauropsida with that of Amphibia. I may add that several authors, as Van Beneden (No. 13), Wenckebach (No. 15), and Will (Nos. 18 and 21) recognise the yolk-plug in this place.

I shall next describe how the invagination-cavity, as described above, comes to open below and becomes united with the large subgerminal cavity in the yolk. This process has, so far as I am aware,

* Contribution I. I shall refer to the papers in the present series of Contributions by their numbers in order of publication. See the list at the end of the present article.

never been treated with the fullness which its importance deserves. A careful study of this process has given me results which, I venture to think, are of the greatest importance in discussing the problem of gastrulation in the secondary meroblastic egg.

The surface views, Figs. 2–5, and the sections, Figs. 13–17 are introduced to illustrate this process. Figs. 2 and 2*a* are of the stage nearest to that represented in Fig. 1. In the dorsal view (Fig. 2), the dorsal opening of the invagination-cavity has now become a narrow crescent-shaped slit with the concavity directed forwards.* In the ventral view, the primitive knob has become larger. Viewed with a low power, the surface of the knob is tolerably smooth, although the figure represents it perhaps as a little too much so. The longitudinal section (Fig. 13) of this embryo shows distinctly that the depth of the primitive knob has grown greater in this stage than in that of Fig. 9. The invagination-cavity has extended itself much deeper and shows distinctly two limbs, one vertical and one horizontal. The roof of the cavity which is as before continuous with the epiblast, shows a distinctly columnar arrangement which is, however, gradually lost both anteriorly and superiorly. In these directions it merges gradually into an irregular network of cells which is in turn continuous with the lower layer of the embryonic shield. As was the case in the former stage, there is again below the primitive knob, no independent sheet of cells continuous with the lower layer of the shield, as described by Wenckebach or Mehnert. On the contrary, this and the succeeding figures (Figs. 14–17) give the impression that the lower layer of the embryonic shield extends below the epiblast

* Will (No. 21. p. 147) says: "Dieselbe (*i.e.* die Urmundspalte) tritt zuerst im vorderen Abschnitt der Primitivplatte auf, und hat zunächst die Form einer Sichel-rinne, nach Schwund der Sichelhörner aber einer rundlichen Delle" That is, his figures 8 and 9 are less advanced than his figures 4 and 10 so far as the shape of the blastopore is concerned. If the first two figures named are comparable to my figures 2, 3, 4, and the latter figures (his figures 4 and 10) to my figure 1, I can not but think that Will is mistaken in his views.

right up to the angle where the epiblast is reflected downwards at the dorsal lip of the blastopore, and that the primitive knob has been capped on to it from below, although now irrevocably fused with it by a protoplasmic network. The floor of the cavity shows two distinct divisions. In the posterior part (the vertical part in the section) there is a compact mass of cells which have evidently been proliferated from the floor of the cavity. This is the posterior median part of the commencing peristomal mesoblast. In the anterior half of floor, the vacuolated network comes very near the cavity, being separated from it only by a thin sheet of cells.

In the next stage (Figs. 3 and 3*a*), we notice one striking change in the ventral surface view of the embryo. While the top of the primitive knob (spoken of with its ventral surface as uppermost, see Fig. 3 *a*) is comparatively smooth as in the former stage, its base has assumed a honey-combed structure and this structure is spreading itself over the ventral surface of the embryonic shield.

Fig. 14 is a longitudinal section near the median line of this embryo. Compared with Fig. 13, the primitive knob has a longer antero-posterior extension and it will be seen that this increase is due almost entirely to the growth of the anterior half. The forward edge of this half is gradually encroaching on the ventral surface of the embryonic shield (*cf.* Fig. 3 *a*) and is thus giving the primitive knob ever greater extension. Wenckebach (No. 15), Will (Nos. 18 & 19), and Mehnert (No. 8) agree in thinking that the forward growth of the primitive knob takes place by its front growing edge insinuating itself between the epiblast and the lower layer of the shield, and quite independently of these two sheets of cells.* My sections do not allow me to

* In his latest paper (No. 21), Will admits that where gastrulation is completed by the formation of the *Kopffortsatz*, the "primary" and "secondary" endoderm cells cannot be clearly distinguished and that the former may grow by addition of the cells of the latter formed *in situ* (p. 48).

come to the same conclusion, as a glance at Figs. 13 and 14 will show. Both the surface-views (Figs. 3*a et seq.*) and the sections give us even an impression that the primitive knob is spreading itself under the lower layer of the embryonic shield. In the parts where the primitive knob has once established itself, we can, however, no longer distinguish cells that have come from the primitive knob from those of the lower layer of the shield: they are indistinguishably fused. The invagination cavity at this stage (Fig. 14) has much greater longitudinal extension than in that of Fig. 13. I can discover neither at this nor at any subsequent stage a posteriorly-directed limb of the invagination-cavity, such as is described by Wenckebach (No. 15) in his Fig. 3.

There is nothing special to say of the roof of the invagination cavity, except that the points described in the previous stage are all more pronounced in this one. In the floor, there are some important changes. In the posterior half, where the mass of the peristomal mesoblast, grown much more compact, is easily recognisable, there is not much that is new. But in the anterior half of the floor, the wall of the invagination cavity is no longer so sharply defined as before, and some meshes of the cellular network in the primitive knob even open into the invagination-cavity, so that we can here, already in this stage, pass by a labyrinth of intercellular passages from the invagination-cavity to the subgerminal yolk-cavity. It should be specially noted that the anterior and of the invagination cavity is distinct and does not share in the dissolution of the anterior part of the floor.

With the growth of the embryo, the changes in progress between the stage of Fig. 13 and that of Fig. 14 become more and more pronounced. The primitive knob grows forwards more and more on the ventral surface of the shield, so that its antero-posterior diameter

is ever getting longer (Figs. 14, 15, 16 & 17). In the anterior part of the floor of the invagination-cavity which was already losing its sharp definition in Fig. 14, the disruption has proceeded one step farther in Fig. 15. In this figure, not only this part of the floor is giving away, but the network of cells lying underneath it, and between it and the subgerminal yolk cavity, has been largely absorbed. In Fig. 16, the process of breaking through is seen to be complete, and the invagination-cavity has now a clear opening below. I think it almost certain that such a clear and comparatively large opening has been produced by the running together of several small openings, such as we see in Figs. 14 and 15, which put the meshes of the cell-network in communication with the invagination-cavity. In fact, in Fig. 16 we can still see several such openings in the floor of the cavity in that part of the network situated behind the large anterior opening and in front of the compact peristomal mesoblast mass. Comparison with Fig. 17 makes it probable that this part of the cell-network is to be eventually absorbed, for the single large opening extends in the latter back almost to the peristomal mesoblast. It should also be noticed in Fig. 16 that the extreme anterior end of the invagination-cavity is clearly recognisable and does not participate in the breaking through, which seems to be confined to the floor. We should therefore remember that although the anterior end may not be recognisable in later stages (*e.g.* Fig. 17), it is the floor which is open below. The surface views of the stage at which the invagination cavity has just opened below are given in Figs. 3 *bis*, 4, and 4*a*. There is considerable difference in the appearance of the two embryos which I am not able to explain. I drew them just as they appeared under the microscope. I am rather inclined to think that Fig. 4*a* represents a more normal appearance, if we are to judge from the succeeding stages, although I am unable to detect anything unusual

in the sections of the other embryo (Fig. 3 *bis*), Fig. 16 in fact being one of them.

From the facts given above, the conclusion is reached that the invagination-cavity comes into communication with the subgerminal yolk-cavity by the absorption of the most anterior part of its floor as well as of the cell-network lying underneath this part. In this view, I find myself in agreement with Mehnert (No. 8, p. 411) who says:—"Wenn der Einstülpungssack etwa die halbe Länge des Embryonalschildes erreicht hat, schwindet in dem vordersten Abschnitte seine untere Wand und das mit derselben innig verwachsene Paraderm, so dass durch diesen Vorgang eine freie Communication zwischen der Einstülpungshöhle und der Subgerminalhöhle gebildet wird" I would only remark that this disappearance does not take place suddenly, as Mehnert's words might possibly lead one to suppose. As my sections show, it is already begun as early as in the stage given in Fig. 14. Wenckebach's Fig. 4 (No. 15 p. 60) is very much like my Fig. 16, except for the differences already specified. Although his views are not given in detail, I think, they are probably similar to mine. Will's views are essentially like mine; only he insists on the greater forward and lateral extension of the invagination-cavity before it opens below. This is especially the case with the tortoise. He (No. 19, p. 191-2) says: "Aus diesen Stadien geht nun die wichtige Thatsache hervor, dass auch der Urdarm der Schildkröte noch in seiner ganzen Ausdehnung hohl ist und dass seine Ausdehnung absolut und relativ diejenige des Gecko noch übertrifft. Während derselbe beim Gecko die vorderen und seitlichen Ränder des Schildes nie vollständig erreicht, nimmt derselbe bei der Schildkröte stets die ganze Fläche des Schildes ein. Der Durchbruch des Urdarms erfolgt auch hier ganz ebenso wie beim Gecko, so dass die Fig.

7 meiner oben zitierten Mittheilung (No. 18 of my list. Same as Fig. 17*b* of No. 21) auch geeignet ist, die Verhältnisse bei der Schildkröte zu illustriren. Es treten zunächst einige wenige isolirte Durchbrechungen der untern Urdarmwand (nebst dem unter derselben wegziehenden Dotterblatt) ein; indem sodann beständig neue Lücken auftreten, die alten sich aber vergrössern gelangt man zu Stadien, bei den von der gesammten unterer Urdarmwand nur noch ein unregelmassiges, bei den verschiedenen Embryonen verschieden gestaltes System von Netzbalken erhalten geblieben ist. Schliesslich kommen auch diese letzten Reste zum Schwunde, wodurch dann das bisherige Urdarmlumen mit dem subembryonalen Raum zusammenfliesst." After seeing his Figs. 55*a* and *b* (No. 21) it seems no longer possible to doubt the great anterior extension of the archenteron in Gecko, before it breaks open. As to this point in Chelonia I shall reserve my judgment until his promised full paper on the tortoise appears.

Van Beneden (No. 13) describes in Mammalia two kinds of openings by which the chorda-canal comes to open below into the blastoderm cavity, viz:—(1) an anterior transverse slit, and (2) several openings which soon run together into a single posterior longitudinal slit. From what has been stated above, I need hardly say that I do not find any such differentiation of openings in Chelonia.

The changes that follow on the breaking through of the invagination-cavity can best be seen in the surface views. Figs. 5 and 5*a*—8 and 8*a* are introduced to illustrate this point. We have seen how the primitive knob, at first confined to a small accumulation of cells at the posterior edge of the embryonic shieid (Fig. 9), gradually spreads itself anteriorly until it comes to occupy quite a considerable area on the ventral surface of the embryonic shield (Figs. 13-16), when the invagination cavity breaks through be-

low (Figs. 4*a* and 16). The part that has been covered by the cells of the primitive knob can be very plainly distinguished on a ventral view of the blastoderm, showing mostly a trabecular network (Fig. 4*a*). This gradual spreading of the cells from the primitive knob over the ventral surface of the embryonic shield is continued long after the breaking through of the invagination-cavity. Figs. 5 and 5*a* are only a little advanced on Figs. 4 and 4*a*. The area of the network is not yet very large, but in the stage next introduced (Figs. 6 and 6*a*) it has expanded itself considerably. A great change is now noticeable: a circular area at its centre shows no longer a network but presents a smooth compact surface. This is produced by a continuation forwards of the process by which cells in the roof of the archenteric-cavity, beginning at the dorsal lip of the blastopore, have gradually assumed the columnar shape and formed themselves into a compact sheet (Figs. 13-18)

In Figs. 7 and 7*a* the spreading of the part derived from the primitive knob has gone one step further. Not only is the area occupied by the network larger but the compact part in the centre is considerably enlarged by its extension anteriorly in the median line. There is also another noteworthy new feature: at the anterior end of the median compact area, there is a slight transverse ridge. T h i s is t h e c o m m e n c i n g h e a d - f o l d. The last stage in which I was able to detect traces of the network is shown in Figs. 8 and 8*a*. At the front end of the embryonic shield, a patch of the network could be faintly traced. I think it almost certain that the part derived from the primitive knob does not extend itself much beyond the area of the embryonic shield, and that it gradually thins itself out and ends by becoming simply continuous with the primitive lower layer at or near the periphery of this area (Figs. 17 & 18). With the exception of the patch above-mentioned, the ventral surface of the embryonic shield

presents now a smooth compact appearance. The head-fold and the chorda-groove have already become conspicuous.

As the head-fold, formed well within the edge of the embryonic shield, marks the anterior end of the embryo, and therefore of the archenteron or the adult alimentary canal exclusive of the stomodæum ; as the primitive knob marks the posterior end of the embryo ; and as the lateral body-wall is formed from the lateral folds, also arisen within the embryonic shield, we are justified in coming to the very important conclusion that the body of the future embryo and consequently the definitive alimentary canal is formed entirely within the area covered ventrally by cells derived from the primitive knob. This speaks in favor of the assumption that the invagination cavity is the archenteron and gives rise to the future alimentary canal. I shall discuss farther on how we ought to regard the breaking through of the invagination-cavity and the gradual spreading of the cells of the primitive knob over the ventral face of the embryonic shield.

The reason why the advancing edge of the primitive knob is marked by a zone of network is probably, I think, that such a structure allows free and easy access of the nutritive liquid of the yolk to the deeper parts of the tissue.

The network such as is here described, has been seen many times before. For instance, Ishikawa and I noticed it in a Trionyx blastoderm (Fig. 1*b* of Contrib. I) without knowing its significance. Again, Fig. 10, Contrib. III represents the same thing in cross-section in an embryo of Clemmys. Mehnert (No. 8) gives beautiful illustrations of stages showing the network, in his Figs. 4-13. He, however, gives an explanation of it which is at utter variance with the one given above, for according to him, it is concerned with the process of the mesoblast formation. He states

that in the anterior part of the embryonic shield, the dorsal roof of the archenteric cavity divides itself into two layers : (1) a lower one consisting of a single layer of low cells representing the definitive hypoblast, and (2) an upper one consisting of stellate branched cells representing the "Rumpf-mesoblast" (his Figs. 22 & 23). In the course of this separation, the dorsal roof which is at first composed of compact columnar cells becomes permeated by vacuoles, and he says that "das im Flächenbilde eruirte Netz der Ausdruck für die aus dem Verbande des Urdarm-epithelhofes (scl. oberer Urdarmwand) losgelösten Mesodermstränge war, welche sich im Furchungsspalte centrifugal zwischen Ektoderm und Paraderm weiter vorschieben" (p. 434). He thus calls the area of the network with the central compact part the "Rumpfmesodermhof." Moreover he makes this process of the mesoblast formation begin at the cranial end and proceed backwards. He also says that "die periphere Ausbreitung des Mesodermhofes nicht im proportionalen Verhältnisse zur Grösse der Area embryonalis (scl. Embryonalschild) steht" (p. 434). Mehnert's views can not be reconciled with mine: one of us is wrong. Except as to the single point that the network grows centrifugally, I am obliged to differ from him in almost every particular. This network has in my opinion nothing to do with the process of the mesoblast formation. My views on the latter process have already been given in great detail in two former papers (Contrib. I. & III.) and I do not intend to go into them again in this paper. The network is simply the surface expression of cells from the primitive knob spreading themselves over the ventral face of the embryonic shield. In what light we ought to regard this process I shall discuss farther on. But whatever it is, it does not begin at the cranial end and proceed backwards. In obtaining materials for the present investigation, I opened on consecutive days a certain number of eggs from the same deposits, and observed

the progress made during the interval of time between the two successive acts of taking out, making of course due allowance for fast or slowly developing eggs. Fig. 1 and Figs. 4–8 with some intermediate stages, which I have not introduced here, belong to one of the series obtained in this manner, and these show, conclusively so my mind at least, that the area of the network spreads itself gradually underneath the embryonic shield from the spot where the invagination-cavity first breaks through, towards the periphery of the thield *i.e.* from the posterior primitive knob anteriorly over the embryonic shield. It is not the network that is gradually encroaching on the central compact area, as Mehnert assumes, but just the reverse; for the central compact area is formed out of the area of the network. The series, Figs. 5*a*—8*a*, shows also that the area of the network increases with the age of the embryonic shield. I can not therefore accept Mehnert's explanation of the appearance of the network on the ventral face of the blastoderm.

As I said just now, I do not propose to go into the mesoblast-formation again in this paper. I would merely remark that in the stage corresponding to Fig. 17, I already see the establishment of the chorda-hypoblast and the stretch of the epithelium on each side of it which becomes transformed into the gastral mesoblast. (Compare Fig. 11, Contrib. III.).

There are two other points on which I wish to make some remarks.

The first of these is in regard to the position of the primitive knob relatively to the embryonic shield. In Fig. 1, the primitive knob lies for the most part outside of the embryonic shield, only about one-third of its antero-posterior extension being within the shield. In Figs. 2 and 3 it is about one-half, and in Fig. 4, entirely within the shield. This is no doubt brought about by the gradual

extension of the epiblastic area composed of columnar cells. In later stages, (Figs. 6, 7, 8), the mass of the peristomal mesoblast no doubt helps in causing opacity in the posterior region. From the stage of Fig. 6 on, the embryonic shield, which has hitherto passed gradually into the surrounding parts, becomes sharply marked off from the circumjacent transparent area, in which it is eccentrically placed, and becomes apparently diminished in size.

The second point on which I wish to touch is as to the dorsal opening of the invagination cavity. In the earliest stage I possess (referred to on p. 236–7), it is a squarish pit rather elongated in the antero-posterior diameter. In Fig. 1, it is a wide open cavity elongated transversely. In Fig. 2, it is a crescent-shaped transverse slit, no longer gaping, and with its concavity turned forwards. The same can be said of Figs. 3 and 4. In Fig. 5, the blastopore is nearly straight across. It has a slight notch in the median line open backwards. In Fig. 6, the ends of the slit-like opening have turned backwards so that now the concavity faces backwards. In further growth, the backward curvature becomes greater and greater, until it becomes a horse-shoe shaped slit, as can be seen in figures contained in the former Contributions. Of the significance of this change of shape I shall speak later on.* The yolk-plug, which can be traced more or less clearly from the first, becomes very distinct as the backward curvature becomes greater, and sticks out between the two limbs of the horse-shoe.

The yolk-plug in Fig. 8 is very peculiar in that it has a groove in the median line. The cross sections of this region also show it to be

* In his latest paper, Will (No. 21) seems to consider the enclosure of the primitive knob within the embryonic shield as intimately connected with the change of the shape of the blastopore-opening. Both are, according to him, due to the forward growth of the yolk-plug and the consequent shoving forwards of the blastopore-opening (see p. 127 *et seq.*). I find myself unable to accept his views.

a deep fissure cutting the yolk-plug into halves. I am unable to give any explanation of this groove, which is very unusual, this being the sole instance among hundreds of chelonian embryos that have passed through my hands. I think it may probably be teratological. Kupffer (No. 6, Taf. IV. Fig. 40 *f.* & *g.*) gives two figures of Coluber that are strikingly like this.

To sum up the facts of Gastrulation as above described :—

1. When segmentation has gone on for some time, there is established in the blastoderm two layers: (*a*) the superficial epiblast composed of columnar cells, and (*b*) the lower layer composed of irregular stellate cells and probably not forming a complete sheet.

2. This separation into two layers takes place in all parts of the blastoderm with the exception of a small area at the posterior end of the future embryo. Here not only is there no differentiation of layers but a thick knob consisting of a network of cells is produced by the accession of cells from the subjacent bed of yolk. The mass can not be said to belong to either of the two layers above named. This is the *Primitive Plate or Primitive Knob.*

3. In the middle* of the *Primitive Knob*, an invagination cavity is produced, which at first goes straight downwards but soon takes a forward horizontal course. This is the *Invagination-Cavity* or the *Archenteron*. Its dorsal opening is the *Blastopore*. The invagination-cavity

* Will (No. 21) is no doubt quite correct in printing out that the invagination-cavity begins much nearer the anterior than the posterior end of the primitive plate. In front of it there is only the future anterior or dorsal lip of the cavity.

extends itself gradually forwards, *pari passu* with the anterior enlargement of the *Primitive Knob*.

4. The roof of this invagination-cavity which becomes continuous with the epiblast of the embryonic shield at the anterior lip of the blastopore, assumes a columnar arrangement, the process beginning at the posterior end and proceeding gradually forwards. Out of the median part of it is established the *Chorda dorsalis*, and from a certain stretch of columnar epithelium on each side of it is developed the gastral mesoblast.

5. The floor of the invagination-cavity is divided into two parts:—(*a*) the posterior which proliferates the peristomal mesoblast, and (*b*) the anterior which losing definiteness is finally absorbed, together with the whole thickness of the cell network placed beneath it, *thus putting the invagination-cavity in communication with the large subgerminal cavity in the yolk.*

6. The primitive knob which was gradually spreading itself over the ventral surface of the embryonic shield before the breaking through of the invagination-cavity continues to do so after that event. It spreads from the spot where the invagination-cavity first broke through away towards the periphery of the shield. Its advance in later stages is marked by a zone of cell-network with a compact central area. When the whole of the ventral surface of the embryonic shield has been covered, the process stops. The cell-network afterwards changes into compact cellular sheets.

7. The head-fold is formed some distance behind the anterior edge of the embryonic shield.

8. The future embryo and consequently the definitive alimentary canal is formed entirely within the area covered ventrally by the part derived from the primitive knob.

Putting the results in another way, they may be summed up as follows :—

From the epiblast of the embryonic shield, THE EPIBLAST and ITS DERIVATIVES of the future animal is derived. In the region of the primitive plate and its anterior enlargement are produced the INVAGINATION-CAVITY (the Archenteron), the YOLK-PLUG, the CHORDA, the MESOBLAST (both peristomal and gastral), and the DEFINITIVE HYPOBLAST and ITS DERIVATIVES. The primitive lower layer forms the wall of the yolk-sac, and contributes to the future animal only in so far as some of its cells are unrecognisably incorporated with the cells of the primitive knob, when the latter spreads itself over the ventral surface of the embryonic shield.

On the last point, I find myself at variance with Wenckebach who makes the cœnogenetic hypoblast take part in the formation of the anterior part of the embryo-body, "namentlich an dem cranialen Wachsthum von Chorda und gastralem Mesoderm (No. 15, p. 76).

Theoretical Considerations.

If we represent the chelonian egg diagrammatically in the light of the facts described in the preceding pages, we shall obtain something like that given in Woodcut I.

Woodcut I.

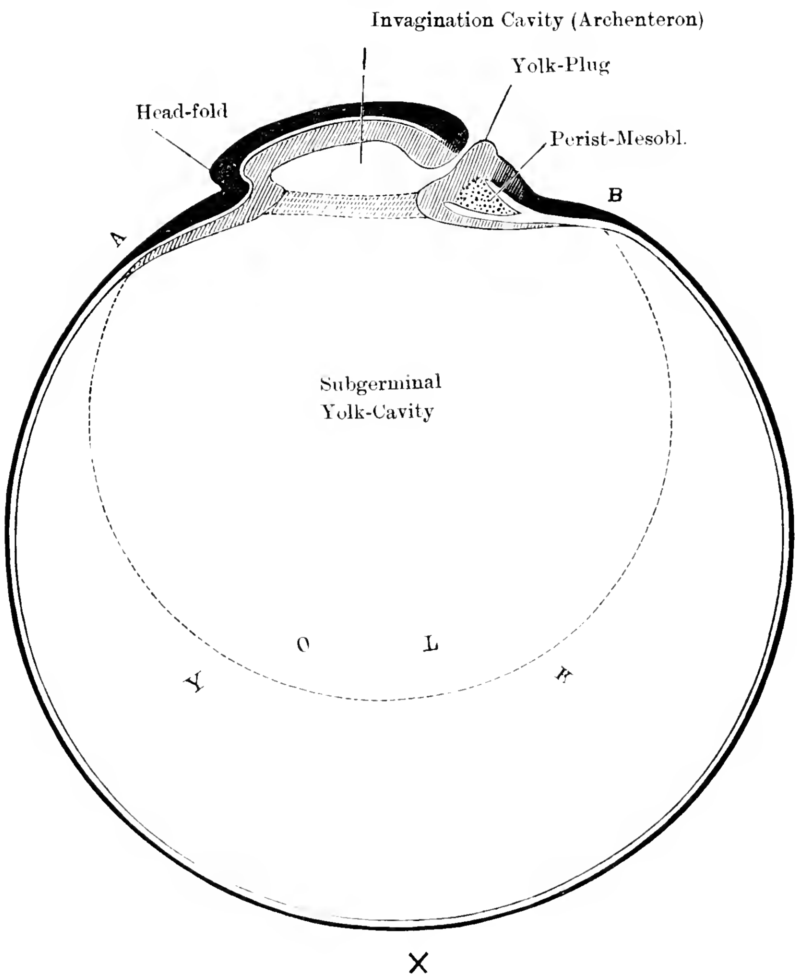
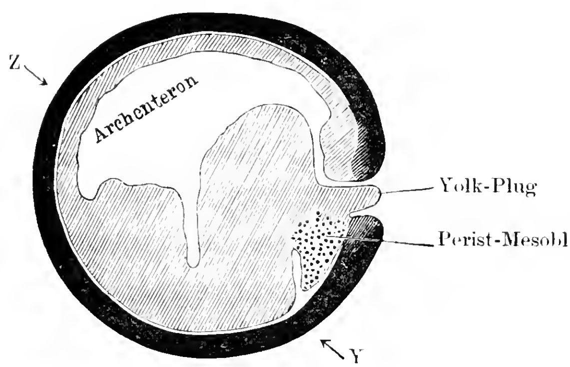


Diagram of a Chelonian Egg.

$A-B$ represents the embryonic shield with the enclosed primitive knob. Within the shield is established the whole of the future embryo. $A-X-B$ is the yolk-bag with the large subgerminal cavity filled with nutritive liquid. The invagination-cavity which has extended forwards *pari possu* with the anterior extension of the primitive

knob has by the absorption of the anterior part of its floor (indicated by dotted lines) been put in communication with the subgerminal cavity in the yolk. The anterior end of the invagination-cavity is clearly recognisable at the time of the breaking through ; it becomes invisible for a time after that event, but is soon marked out again by the commencing head-fold. The thick part of the hypoblast (marked with slant lines) is intended to show the extent to which cells from the primitive knob spread themselves. The structures behind the invagination cavity—the yolk-plug, the peristomal mesoblast—have been fully described in Contribs. I. & III.

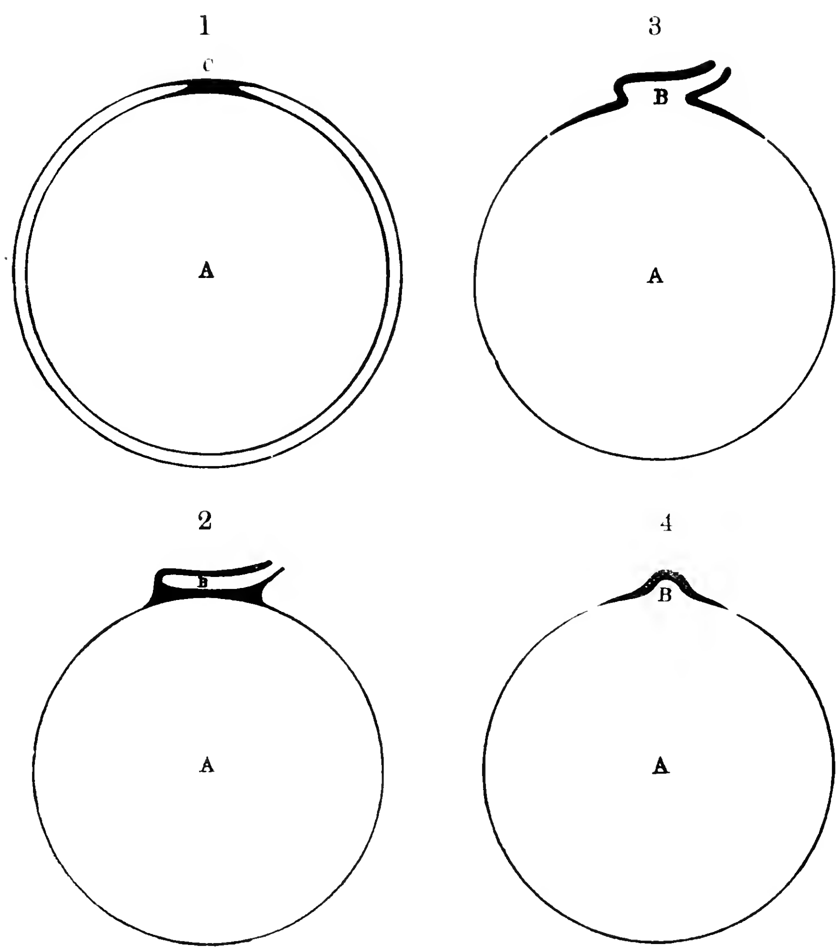
Woodcut II.



When we compare this diagram with the well-known one of an amphibian egg (Woodcut II.) given by Hertwig, their similarity becomes very striking. The structures dorsal to the line $Z-Y$ in the amphibian egg can be identified, part for part, on the embryonic shield of the chelonian egg. In homologizing these two eggs a great deal depends upon the view we take as to the nature of the invagination-cavity, the breaking through of the same, and the large subgerminal cavity into which the the invagination cavity opens. I have already made mention of the assumption that the invagination cavity gives rise to the definitive alimentary canal (p. 245). The following considerations will make my views clear and I trust, justify them at the same time :—There is in the chelonian egg a large yolk reservoir A (Woodcut III. 1).

This, let us suppose, is surrounded by a layer of cells (although in point of fact the lower pole is not enclosed until a much later period), except at the point *C* where there is a mass of cells in which both the epiblast and the layer surrounding the yolk are merged. This is the Primitive Plate or Knob. In this knob, there arises an invagination, *B* (Woodcut III. 2) which grows forwards together with the anterior elongation of the primitive knob. Assume

Woodcut III.



for the present the invagination-cavity (*B*) to be the **Archenteron**. Then the yolk reservoir *A* must from the nature of the thing be an appendage of the invagination-cavity *B*. But owing to its enormous size compared with *B*, the bodily invagination of the yolk is out of the question. It forms the most conspicuous part of the egg from the first, and begins to surround itself with a cell-layer long before the invagination-cavity *B* makes its appearance even. Hence, *A* can only secondarily come into connection with *B*. This happens by the anterior part of the floor of *B* flaring out, so to speak, into a funnel-shaped opening. This is the meaning of the breaking through of the invagination-cavity. The spreading of the cells derived from the primitive knob over the ventral surface of the embryonic shield after the breaking through of the invagination-cavity may be regarded as the gradually thinning wall of the funnel-shaped opening making itself continuous with the cell-layer surrounding the yolk (Woodcut III, 3–longitudinal section, 4–cross section). When the alimentary canal is formed definitely, later on in the course of development, the wall of the funnel-shaped opening of the invagination cavity is again tucked in as the splanchnopleura.

The above course of reasoning explains all the events accompanying the invagination and thus justifies the assumption that the invagination-cavity (*B*) is the Archenteron corresponding to the part marked as such in the amphibian egg (Woodcut II.) and the whole yolk-bag must be regarded simply as a part of its ventral wall that has become bulged out on account of the enormous accumulation of nutritive matter within it. The presence of a large subgerminal cavity in the yolk filled with a nutritive liquid is a physiological accident, so to speak. I agree with Van Beneden, Keibel, and Wenckebach in regarding it as intercellular space in the yolk. It is a cavity arisen solely from physiological necessity

and having a comparatively insignificant morphological value. Although the whole yolk-sac should be regarded as a diverticulum of the archenteron and although it has a definite morphological value, it is a matter of comparative indifference, so far as morphology is concerned, whether its inside is filled with cells charged with yolk-granules, or with free yolk-spheres, or with a nutritive liquid or with a mixture of all three. Looked at in this light, the chelonian egg is nothing but the amphibian egg, with an enormous ventral saccular appendage surcharged with nutritive matter.* The whole yolk-sac (Woodcut I. $A-X-B$) must not, however, be looked on as strictly homologous with the part of the amphibian egg ventral to the line $Z-Y$. For, in the latter, the epiblast of that part becomes the ventral abdominal wall of the future animal, while in Chelonia the epiblast of the yolk-bag becomes later a part of the serous envelope,—the ventral abdominal wall of the embryo being formed within the embryonic shield above the yolk-sac, and the yolk-sac with the enclosing sheet of hypoblast and mesoblast cells migrating within the body of the embryo. When it has done so, nobody has any difficulty in accepting it as an appendage of the alimentary canal which has for its function the storage of nutritive matter. My contention is that as such it should be looked on from the first. The archenteron is at first so utterly insignificant in size compared with the yolk-sac that the true nature of the latter is obscured: none the less the yolk sac is a mere appendage of the archenteron. This view makes it necessary to regard the primitive lower layer enclosing the yolk-sac as a part of the

* It will be seen that further consideration has made me modify in some details my views as set forth in the preliminary notice sent to the *Anatomischer Anzeiger*, and published in that journal, Nos. 12 & 13, 1893..

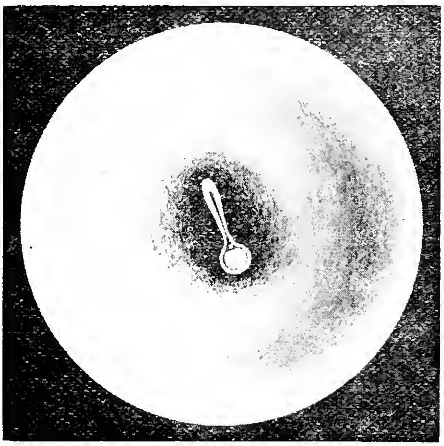
hypoblast. That it arises before the invagination of the archenteron can be explained by the principle of precocious segregation, as has been pointed out by Hubrecht (No. 5). The name "cœnogenetic hypoblast" which he applies to this layer, seems therefore very appropriate as the part derived by invagination may be called the "palingenetic hypoblast."

I think the objection on the part of Robinson and Assheton (No. 10) that the yolk-plug can not be present at the spot designated by Ishikawa and myself is fully answered by comparing the two diagrams (Woodcuts I. & II.). The yolk-plug can not only properly be found at this spot but its presence here is one of the significant, although not the essential, features in homologizing it with the amphibian egg. This is an example of those cases where a secondary characteristic is of great service in identification.

According to the views set forth above, the enormous accumulation of yolk has profoundly affected the course of development in the chelonian egg, especially in the precocious development of a part of the hypoblast, and in the rapid spreading of the blastoderm over the surface of the egg. There is left, however, in the centre of the blastoderm a certain amount of raw undifferentiated materials in the shape of the primitive plate or knob in order to go with it through certain developmental processes of palingenetic character :—the invagination of the archenteron with the consequent establishment of the chorda-hypoblast, the peristomal and gastral mesoblast, the yolk-plug, and the definitive hypoblast. The changes in the shape of the blastopore from a crescent with its concavity turned anteriorly to that of a horse-shoe with its two limbs directed backwards and enclosing the yolk-plug between them must be looked on as the remnant of that process by which the epiblast gradually encloses the endoderm cells in the amphibian ovum or the

yolk in the Elasmobranch egg. If we make a companion diagram to the well-known series given by Balfour (Comp. Embryol. vol. II. Fig. 175) it will be like Woodcut IV. The upper pole of the egg in Reptilia is capped by a small patch where nearly all the changes which in Amphibia are gone through by the whole egg, are performed. Accordingly, the enclosure of the yolk by the blastoderm in the chelonian egg is of a very different nature from the enclosure of the yolk in the Elasmobranch, for while the former is a simple growth of the edge of the blastoderm, and of cœnogenetic character, the latter is a part of the process of invagination and of palingenetic character. That the yolk in Chelonia is not completely enclosed till the embryo has made much progress is due to its large size, and may be regarded as of quite secondary significance. I thus find myself obliged to put aside the yolk-blastopore of Balfour as no longer tenable in Sauropsida.

Woodcut IV.



After what has been given above, I need hardly say that I accept the views of Rabl (No. 9) as to the loss and acquisition of the yolk in vertebrate eggs several times in the course of the phyletic development. All the facts given above tend to prove that Chelonia possesses a secondary meroblastic ovum in contrast to the primary meroblastic ovum of the Selachians.

My views overlap more or less those of previous writers, such as Wenckebach (No. 15), Will (Nos. 18, 19 & 21), Mehnert (No. 8) and Rabl (No. 9). It would, however, be a tedious and useless task

to go over the writings of these authors and point out wherein we agree or differ. The reader acquainted with the literature will be able to do this for himself. The points which I want specially to emphasize are however as follows :—

1. The PRIMITIVE PLATE or KNOB is raw-material left at the centre of the blastoderm, by means of which certain palingenetic processes are gone through.

2. The INVAGINATION-CAVITY is the ARCHENTERON, and gives rise to the alimentary canal and the organs derived from it exclusive of the proctodæum and the stomodæum.

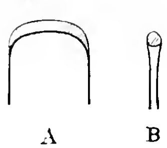
3. The YOLK-SAC must be regarded as a ventral appendage or diverticulum of the ARCHENTERON in which nutritive matter is stored in solid or liquid form.

4. Owing to the enormous size of the YOLK-SAC, it and the ARCHENTERON are formed separately from each other, and come only secondarily into connection.

Having considered the chelonian egg in its relations with that of Ichthyopsida let us now see how it compares with the avian or mammalian egg.

If the process by which the blastopore in Chelonia has assumed a horse-shoe shape (Woodcut V. A) continues on after the state A is reached, as actually happens in Amphibia (See Figs. 18 & 19, No. 10), the lateral lips will coalesce* and there will result the primitive streak of the avian egg (B). In cases where the lips have not quite coalesced, we should expect to find the yolk-plug sticking out

Woodcut V.



* I am gratified to find this verified within the group of Reptilia. Will (No. 21) has found in Gecko that the lips of the blastopore approach each other very closely and form a primitive streak.

between them, and such is actually the case as seen in Figs. 15 and 32 of Duval (No. 3). The annexed woodcut will also explain why the posterior limit of the primitive streak is not as sharply defined as the anterior. This view makes it plain that the homologue of the primitive streak in Chelonia is the lips of the blastopore which are, however, still so wide apart from each other that the name "streak" is hardly applicable to it. It should be noted that the primitive plate or knob is not the homologue of the primitive streak. The latter has potentially in it not only that but a great deal more. It is in fact a mass of raw undifferentiated material from which various structures are produced. This view also makes it evident that as the primitive streak is almost the first feature visible in the development of the avian blastoderm, a great many changes of palingenetic character observed in the chelonian egg before the establishment of what corresponds to the primitive streak, are necessarily skipped over in Aves, which are therefore not very good subjects in which to study the process of gastrulation. The removal of the primitive streak to the centre of the blastoderm must also be explained in the way I have indicated above in the case of Chelonia.

Comparison of the reptilian ovum with the mammalian seems easier. The facts given in this paper agree, with the exception of some minor details, very closely with those communicated in Van Beneden's preliminary notice (No. 13). It seems to me that the primitive streak and Hensen's knob together correspond to the primitive plate of Chelonia, and the "Kopffortsatz" to the forward growth of the primitive plate. I can not, however, accept Van Beneden's theory of "Lecitophor" and "Blastophor." Exactly what I cannot accept lies in the emphasized words of the following quotation:—"Wenn diese Auseinandersetzungen richtig sind, wie ich es glaube, so ist es klar dass das sogenannte zweiblätterige Stadium

der Säugethiere der Gastrulation d. h. der Einstülpung, die man von der Epibolie auseinanderhalten muss, vorangeht, und dass die zwei Schichten respektiv dem Ektoderm und dem Entoderm des Amphioxus nicht entsprechen. Dieser Schluss geht schön daraus hervor, dass nicht allein die Organe des Epiblastes, sondern auch die Chorda und der ganze Mesoblast aus der äussern Schicht sich bilden." According to my views, the epiblast of the Amniota is homologous with the epiblast of Amphioxus. The difficulty which keeps Van Beneden from accepting this idea lies in this, that not having for comparison the comparatively simple story of the reptilian development he has reckoned as epiblast what corresponds to the primitive plate of Reptilia. If he had recognised the structure which, as I have shown above, can not be said to belong to either layer and then considered the lower layer as precociously developed hypoblast, the conclusion would have been inevitable that the outer layer corresponds to the epiblast of Amphioxus. Keibel (No. 7) has also shown to what contradiction Van Beneden's theory of the "Blastophor" and "Lecitophor" leads. I think, however, I have now removed the second objection of Keibel:—"Dazu kommt dann ferner, dass uns Van Beneden den Beweis dafür durchaus schuldig geblieben, dass nun wirklich die untere Schicht des zweischichtigen Säugethierkeimes und die Keimhöhle desselben mit der Bildung des definitiven Darms der Säuger nichts zu thun hat." I think, the fact that the whole yolk-sac with the subgerminal cavity within it does not form in Chelonia any permanent part of the alimentary canal, makes it highly probable that the same is also true of the homologous structure in Mammalia. As I have more than once stated above, I accept Hubrecht's view of precocious segregation. In many respects my views are very much like his, but I do not think, he makes a clear distinction between the Archenteron and the yolk-sac. Nor do I know from personal

observation whether such a distinction is possible in Mammalia. I am only inclined to think that, since the reptilian and mammalian eggs are alike in so many points, what is true in the former as regards the development of the alimentary canal will in the main be found true also in the latter. I can also find in Chelonia nothing corresponding to his "proto-chordal plate." As to whether there is such an annular zone of hypoblast as he describes which gives rise to the mesoblast I wish to express no opinion. That the "Rumpfmesoblast" arises entirely within the embryonic shield from the materials derived from the primitive knob I hope to have made at least probable in the preceding pages, but whether some temporary mesoblastic structures of the embryo may not arise in Reptilia from such an annular zone as he describes, I am not in a position either to affirm or to deny.

Postscript.

The foregoing article was nearly finished in January of this year. I made an extract of it in the early part of that month and sent it to the *Anatomischer Anzeiger* as a preliminary notice.* As I was giving final touches to the article I received from Dr. Ludwig Will an article of his own entitled "*Die Anlage der Keimblätter beim Gecko*" (*Zool. Jahrbücher; Abth. f. Anat. u. Ont., VI Band, 1 Heft*). As I mention in a previous page, I was not under the circumstances able to make full use of Dr. Will's paper, but inserted remarks on it mostly in footnotes. The foregoing article has since then been lying ready for the press, but its publication was greatly delayed, owing to various extraneous circumstances. When it was at last to be put in the printer's hands, I received a second article by Dr. Will: "*Die Anlage der Keimblätter bei der menoquinischen Sumpfschildkröte*" (*Zool. Jahrbücher: Abth. f. Anat. u. Ont., VI Band, 3 u. 4 Heft*). As it is too late to go over my article again in the light of the facts brought out by Dr. Will, I have decided to add here as a postscript a few remarks on Dr. Will's two papers, as well as on some other articles which have appeared recently.

Will's observations on the two species, Platydactylus facetanus, *Schreib.* and Cistudo lutaria, *Gesn.* coincide throughout. They, I am glad to see, agree also in many essential points with the results I have brought out in this and previous contributions. There are however, several points on which we differ and some of these, it must be confessed, are by no means insignificant.

1. According to Will, a stage in which a sickle is present precedes the establishment of the primitive plate in both the species. Since receiv-

* Published in *Anat. Anz., VIII Jahrg., No. 12/13.*

ing his second article, I have again gone through the chelonian embryos in my possession in order to examine this point. In *Chelonia caouana*, the two youngest embryos which I possess (referred to on p. 236) are not probably much older than that corresponding to Will's figs. 1 and 13 (II Art.) but neither in the sketches I had made of surface views, nor in the sections, was I able to detect any structure resembling the sickle. In *Trionyx*, I was not more successful. But in *Clemmys japonica*, some embryos which I had taken out of the oviduct showed a structure which on surface views looked very much like a sickle.

The annexed figure (Fig. A.) represents one of these in which the sickle is seen to extend to the sides more than in the others. This stage is more advanced than that in which Will figures a sickle, (Fig. 1, II Art.) inasmuch as the invagination cavity has already broken through below. On cutting sections of this embryo, the sickle was found to be due to an accumulation of the lower layer cells continuous with the primitive plate. (See Fig. B.). The epiblast is sharply marked off from this mass, so that it can not be regarded as a part of the primitive plate—at least not in this stage. After becoming familiar

Fig. A.

Ventral View of a Clemmys Embryo taken from the Oviduct.

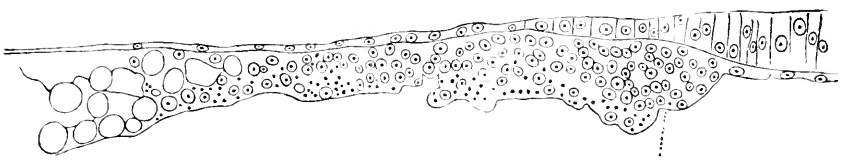


Fig. B.

Fig. B. Posterior part of a longitudinal section of the Embryo given in Fig. A.

with the appearance of the sickle in this series of sections, I was able to detect the same structure persisting in the sections of some older embryos of Clemmys. There is a great deal of variation in the degree of development to which this structure attains in different individuals as well as on the two sides of the same individual. It seems to disappear entirely later. As the mesoblast develops afterwards quite independently of this, it is not what Will calls Kupffer's sickle. For the present, I think, it corresponds probably to the sickle (Koller's sickle) which Will describes in the earliest stage, although there are some features of it which I do not yet quite comprehend and which may finally establish its difference from Koller's sickle.

Apart from the structure which I have described, I can detect nothing comparable to Koller's sickle in my materials.

2. Will makes out a sharp line of demarkation between the ectoblast and entoblast at the edge of the primitive plate (I Art. Figs. 43, 44, and others ; II Art. Figs. 13 *a.* and *b.*). Since reading Will's second article, I have again carefully gone over the sections of my earliest stages, but I am unable to make out such a line at all. As this line is figured in Will's papers as persisting to quite late stages, I am surprised that I do not see it at least in some of my sections, if it really exists.

3. Perhaps the most serious point of difference in the observations of Will and of myself is in regard to the extension of the invagination cavity, before it breaks through below. In Cistudo, Will states that

the invagination cavity becomes exactly co-extensive with the epiblastic embryonic shield, before that act takes place (II Art. figs. 6*b*, 7*b*, 8*b*). In Platydactylus it is said to be only slightly less. In Chelonia caouana, which I have studied, the invagination cavity breaks through, as I have stated in the foregoing article, when it is quite small compared with the epiblastic shield. Will accounts for this discrepancy by assuming that Cistudo and Platydactylus on the one hand and Chelonia on the other are really different in this respect. (*II Art., Nachschrift.* Also in a note "*Ü. d. Gastrulation v. Cistudo u. Chelonia,*" *Anat. Anz., VIII Jahrg., No. 18/19*). In Trionyx, I possess several embryos which are like Fig. 1*b*. of Contrib. I. or Fig. 6 of the foregoing article, so that I think I am justified in concluding that Trionyx is like Chelonia in this respect. In Clemmys there seems to be individual variations as to this point. For instance, if we compare Fig. A. in this postscript with that given in Fig. 1 of my Contrib. III. we find that in the latter, the invagination cavity must have advanced farther forwards, nearer the anterior end of the embryonic shield than the former. So that it is an actual fact that there are variations in different species or within the same species in the proportion of the invagination cavity to the shield. For the present, I am therefore willing to accept Will's assumption as the correct explanation of the disagreement between his statements and mine. And yet I can not help having some doubts lurking in my mind that his Figs. 6*b*, 7*b*, and 8*b* (II Art.) are expressions of something other than the breaking through of the invagination cavity. That in Cistudo the invagination cavity becomes both in length and breadth exactly coextensive with the embryonic shield—not one whit more or less—seems to me very extraordinary. The figures Will copies from Clarke do not certainly show the lateral extension of the in-

vagination cavity to be equal to the width of the shield. In this connection we must remember another fact which Will brings out in another place and which I believe myself able to corroborate, that "die gesammte dorsal Urdarmwand zur Bildung der Chorda und des gastralen Mesoderms aufgebraucht wird" (II Art. p. 612).* If the invagination cavity becomes, as Will maintains, really coextensive with the embryonic shield, it follows from the above-mentioned fact that the lower layer covering the entire ventral surface of the embryonic shield is used up for the above-mentioned purpose and the gut-hypoblast (Darm-Entoblast) must come from outside the shield. But this can not be reconciled with the fact which I have brought out in the foregoing paper and of which there can not be any doubt, that the gut-hypoblast comes from the cells derived from the primitive plate and arises within the embryonic shield. It can not be urged that there are actual differences in this respect between Cistudo which Will has studied and Chelonia which I have studied, for his Fig. 9 (II Art.) is very much like my Figs. 6*a* or 7*a* in the foregoing article, and shows, beyond a shadow of doubt, that in Cistudo as well as Chelonia the gut-hypoblast arises within the embryonic shield. These considerations force me to suspect that possibly there is no great difference in the actual facts between Cistudo and Chelonia in this matter.

4. As I have stated in the foregoing article, I was unable to detect in Chelonia any differentiation of the primary and secondary endoderm such as Will and several others describe.

5. In one place Will does me injustice. On p. 587 (II Art.) he says: "Die Bedeutung dieses Flächenbildes, von dem ich in Holzschnitt Fig. 7. (my Fig. 2, Contrib. III.) eine einfache Skizze

* I am indebted to Dr. Will for pointing out the inaccuracy of my expression on this point under Heading 4 of my preliminary notice (*Anat. Anz., VIII. Jahrg., No. 12/13*).

gebe, konnte von unserm Autor (i.e. by Mitsukuri) nicht erkannt und auch nicht interpretirt werden, weil demselben damals die ähnlichen Oberflächenbilder vom Gecko noch nicht bekannt waren, die allein dieses vereinzelt stehende Bild deutbar machten. Wir erkennen in die Skizze zwei in ihrem hintern Abschnitt nahezu parallel der Mittellinie verlaufende Linien, welche vorn plötzlich stark divergiren. Ich kann dieselben nur als die Insertionsgrenzen des gastralen Mesoderms ansehen." That I was aware of the significance of my figure referred to above is shown by the following words in my Contrib. III (p. 46). "This inward extension* of the gut-hypoblast is probably the cause of the grooves converging posteriorly into the single median chorda groove seen in the surface view Figs 2 and *3a*." If these words are read in connection with what precedes and follows, I think, it will be plain that I had in my mind the significance of this figure to which Will refers above. I am, however, willing to admit that Will has made this point very clear and his fig. E. (II Art. p. 586 or fig. 4, I Art. p. 94) is certainly a very suggestive one.

6. I may perhaps be allowed to make remark on a part of Will's observation on Cistudo. In his second article (p. 542) he says: "Während der Embryonalschild bisher noch vollkommen im Niveau der übrigen Keimscheibe lag, tritt dieselbe auf diesem Entwicklungsstadium zuerst als deutliche, wohl umschriebene Erhebung von herzförmiger Gestalt aus der Keimscheibenoberfläche hervor. Dementsprechend macht sich diese Wölbung an der Dotterseite (Fig. *3b*.) durch eine leichte Concavität bemerkbar." In another place (p. 568), he is surprised that Mehnert's embryos are not more vaulted or bulged out. When Ishikawa and I first undertook the study of Trionyx, we used to open the shell and try to cut the blastoderm out as is usually

* Perhaps the words I used were not entirely happy. If I had said the "inward movement," it would have expressed my meaning more clearly.

done in taking out chick-embryos. At that time, we used to find Trionyx embryos vaulted dorsally, just as Will describes in the quotation given above. Since adopting the method given in the foregoing article of preserving embryos stretched in their natural condition, I have never found the shields vaulted in this manner at any stage: they were always on a level with the rest of the blastoderm.

7. In a note entitled "On Mesoblast Formation in Gecko," (*Anat. Anz.*) No. 12 u. 13, 1893), I ventured to criticize Will's views on the mesoblast formation of Reptiles. Will replies to my criticism in the postscript to his second article, and also in a note "Zur Frage nach der Entstehung des gastralen Mesoderms bei Reptilien" in the *Anatomischer Anzeiger* No. 20, 1893, which has just come to my hand. I must refer the reader to Will's original papers as well as to the above note for his views. Suffice it to say here that Will considers the gastral mesoblast to be formed by a fold which arises in the outer wall of the archenteron and grows towards the median line, thus cutting off the dorsal portion of the archenteron from its lower main portion. The wall of the small dorsal portion thus cut off is said to become the mesoblast. This is put forth in opposition to the view which was first propounded by Hertwig and which appears true to me, viz: that the mesoblast is formed from two diverticula of the archenteron arising directly on each side of the chorda.

Will considers that Fig. 23 of my Contrib. III. which shows a distinct diverticulum on each side of the chorda can not be held to prove the "Divertikelbildung" as it comes from an old embryo which has the chorda already cut off in the middle dorsal region. In his own words*: "Hier sieht man thatsächlich rechts und links neben der Chorda ein kurzes Divertikel, von dem die solide Mesoblastmasse ausgeht, jedoch lässt sich an einem solchen Bild aus dem Ende des

* The note above referred to. *Anat. Anz. No. 20.* p. 681.

ganzen Processes natürlich nicht mehr erschliessen, ob es sich um echte Divertikelbildung oder um Unterwachsung von Seiten der Urdarmfalten handelt, ob das Divertikel das Primäre und die solide Mesoblastmasse das Secundäre ist, oder umgekehrt." Now, the fact is familiar to every embryologist that at a given stage of development a structure, one part of which is already finished may show at another portion of its length only the commencing phases of the process of formation, so that one can see in one and the same specimen the whole process from the beginning to the end. Such seems to me to be the case with the mesoblast in the Clemmys embryo from which my figure 23 is taken. The fact that the mesoblast formation is complete and the chorda is cut off in the middle dorsal region, does not necessarily vitiate what is seen in the head region: here the process of the mesoblast formation is in a less advanced phase, and if a diverticulum is seen there, it is highly probable that a diverticulum is a feature of the mesoblast formation. That there is no such distinct diverticulum seen earlier in the middle dorsal region is because the epiblast presses closely down, and there is no space for the diverticula to curve upwards to any large extent as in the head region. I think, I have sufficiently demonstrated in my Contrib. III, that the diverticulum in the head region corresponds to that part of the primitive hypoblast in the dorsal region which Will calls the "Zwischenplatte," and that this must therefore be regarded as a shallow diverticulum.

Will also objects to my views on the following grounds: "Bei der Auffassung der Zwischenplatte als ein gestrecktes Divertikel müsste der solide Teil des gastralen Mesoderms (*mgr.* in Fig. 1 B.) nicht an dem Rande der Zwischenplatte inserirt. sondern aus der Mitte der letzteren hervorgewuchert sein." (*Anat. Anz. No.* 20. 1893. p. 681) Again "Wäre die Zwischenplatte ein abgeflachtes Mesodermdivertikel, so musste aus ihr sowohl der somatische wie die splanchnische Mesoblast

hervorgehen." (*Ibid*) The first of these objections occurred to me, while writing my Contrib. III. If one examines the cross-sections of Amphioxus as given, for instance, in Hatschek's Taf. IX. (*Studien ü. Entw. d. Amphioxus. Arb. a. d. Zool. Inst, Wien. Bd. IV.* See also Hertwig's Lehrbuch fig. 72), we shall find that the mesoblast pouch spreads ventrally and laterally not from what corresponds to the apex of the earlier diverticulum, but from its outer or lateral wall. The same thing takes place in Reptiles. Although I did not express this distinctly, it was present in my mind, as a reference to the middle of p. 41 (Contrib. III.) will show. These two objections on the part of Will are, I think, fully answered by this consideration.

Will again says: "Die Urdarmfalte würde bei der Mitsukuri'schen Auffassung überhaupt belanglos für die Mesodermbildung und deshalb unverständlich sein" (loc. cit. p. 681). I do not quite see the force of this objection. A fold is needed to mark the outer limit of the diverticulum,* and when the diverticulum is finally to be cut off from the main portion of the archenteron, it takes place by this fold advancing towards the median chorda. I described this inward movement of the fold† in my contrib. III. (pp. 42 and 46; also Figs. 16-17). It is this last phase of the mesoblast formation which Will emphasizes above all others, and on which he builds what he considers to be a new theory of the mesoblast formation (I Art. p. 102). Even in his own views the part of the mesoblast which is formed by "Septenbildung" is only a small proximal portion near the chorda, for the part *mgr* in his fig. 1 B. C. D. (*Anat. Anz.* No. 20, 1893) is according to himself not formed by "Septenbildung" but proliferated from the archenteric

* In a sentence similar to the above, in my note "On the Mesoblast Formation in Gecko" (*Anat. Anz.* No. 12-13, 1893) the word "mark" is by a most unfortunate oversight in proof-reading misprinted "snack"—a mistake which makes my sentence well nigh incomprehensible.

† I admit that I used then the expression "gut-hypoblast"—instead of the word "fold" which I ought to have adopted. But a reference to figs. 16-17 will show that as a matter of fact I had observed a *fold*.

wall. This course of reasoning reduces Will's views practically to the same thing as mine as given in Contrib. III. with the exception of the single point that I consider the "Zwischenplatte" as a flattened diverticulum, while he does not. I have already urged above the reasons for my views, so that I will not again go into them. I must refer the reader to it as well as to my Contrib. III. Notwithstanding that Will says, I have fallen into a fundamental error in confounding "Septenbildung" and "Divertikelbildung," I still think, I was not without reason, when I said in my note (*Anat. Anz.* No. 12 and 13, 1893, p. 434), that "* * * whether the presence of the fold is emphasized or the diverticulum is pointed out as the essential feature does not alter the facts of the case much. Will's objection to Hertwig's theory may therefore be only an apparent one." The difference between "Septenbildung" and "Divertikelbildung" which Will points out is exactly like that between the process of budding and of division. It is not possible to draw a hard and fast line in one case as in the other.

Finally I would like to add that while Will and myself agree as to the essential features of the reptilian development, the above discussion shows that on many minor points we must for the present "agree to disagree," (as I heard the late Prof. Balfour remark on a similar occasion), until fresh observations bring out new facts and enable us to settle these vexing points.

I have very recently received through the kindness of the author, Keibel's "Studien zur Entwicklungsgeschichte des Schweines." (Morphologische Arbeiten, III). It would perhaps be going out of my way too far to offer any extensive remarks on this article interesting though it is to me. The foregoing paper shows that, like himself, I divide the gastrulation into two phases, but these two

phases are different in his case and mine. My own views are (1) that the cœnogenetic hypoblast is formed by precocious segregation, and (2) that the definitive hypoblast is produced by the formation of the invagination cavity which gives rise to the definitive alimentary tract as well as to the chorda and the gastral mesoblast. According to Keibel, "In der ersten dieser Gastrulationsphasen wird bei den Säugern das Entoderm des Darmes und des Dottersacks gebildet, in der zweiten Mesoderm und Chorda" (*loc. cit*, p. 108). That is, the second phase which corresponds to the formation of the invagination cavity in Chelonia gives rise simply to the mesoblast and chorda and has nothing to do with the formation of the alimentary canal. On the latter structure he says, quoting from an earlier work. "* * * so habe ich doch wohl festgestellet. 'dass,' so sagte ich damals. 'wir das Homologon des Urdarmes unter der zweiten Schicht des zweiblättrigen Säugethierkeimes zu suchen haben. Doch wurde demselben nicht die ganze Höhle des Bläschens entsprechen, sondern nur ein ideeller Spaltraum zwischen der unteren Keimschicht und dem Inhalt des bläschenförmigen Keimes, welchen Inhalt ich dem Dotter homologisiren möchte. Die untere Keimschicht des zweiblättrigen Säugethierkeimes entspricht aber nicht dem gesammten Urdarmepithel des Amphioxus, sondern nur den Theil desselben, welche zum definitiven Darm werden. * * *." (pp. 1—2) Lwoff* has also come to a somewhat similar conclusion. I would not like to be understood as opposing this view in a dogmatic spirit. On the contrary, I think, there are several points which seem to favour such an interpretation. For instance, when the invagination cavity breaks open below in Reptiles, the dorsal wall, which alone remains, gives rise only to the chorda and the gastral mesoblast, as Will points out, and if we looked

* Basilius Lwoff:—Ü. d. Keimblätterbildung bei den Wirbeltieren. *Biologisches Centralblatt, XIII. Band. No. 2 & 3.*

simply at such a figure as Fig. 7*a* of the foregoing article, we might naturally come to the conclusion that the invagination cavity gives rise only to the gastral mesoblast and the chorda, and has nothing to do with the formation of the definitive alimentary tract. But we should always remember a fact which I hope to have proved conclusively in the foregoing article that the bottom of the invagination cavity has been removed. When the bottom is still present, we may

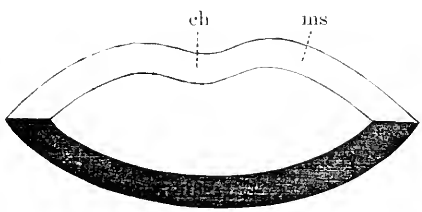


Fig. C.

Diagrammatic Cross-section of the Invagination-Cavity in Chelonia.

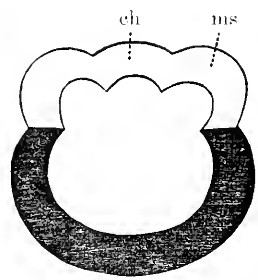


Fig. D.

Diagrammatic Cross-section of the Archenteron of Amphioxus. (*After Götte, from Born. Ergebn. d. Anat. u. Entw. I Bd. p. 494.*)

represent the cross-section of the invagination cavity as in Fig. C. The dark portion is the part that is removed. When we compare it with a homologous section of Amphioxus (Fig. D.) we are struck with their similarity and I think, we are justified in concluding that the invagination cavity of the reptilian ovum is homologous with the archenteron of Amphioxus, and has potentially present in it not only the chorda and the mesoblast but also the definitive alimentary tract. The removal of the bottom, or that part which represents the definitive hypoblast, must in any case be regarded as a secondary process, and can not stand in the way of homologizing the two structures. These considerations, together with the reasons which I have brought out in the foregoing article, incline me more towards those views which I have set forth in the preceding pages than to those of Keibel and of Lwoff.

On another point I would like to say a few words. Referring to the structure in Mammalia which Van Beneden homologizes with the yolk-plug in Amphibia, Keibel says :—" Der Dotterpfropf hat bei den Amphibien keine grosse morphologische Bedeutung, er ist ein Entwicklungs*hinderniss*, er hat kein Funktion. Warum sollte gerade diese Bildung so zäh festgehalten werden, während doch so vieles Andere, das von ungleich grösserer Bedeutung ist, undeutlich wird und verschwindet." As I have insisted on the presence of the yolk-plug in Reptilia, ever since Ishikawa and I first discovered it, I may perhaps give my own view on Keibel's objection. I have in a previous page tried to explain the change of shape in the blastopore in Reptilia as repeating that process by which the lower half of the Amphibian egg becomes enclosed by the epiblast. Taking this in connection with the presence of the yolk-plug, I think that what is inherited is not simply the yolk-plug but the whole process of the epibolic invagination which is gone through in the region of the primitive plate. This is certainly important enough to persist for a long time.

Tōkyō, Oct. 1893.

Contributions to the Embryology of Reptilia.

I. K. MITSUKURI and C. ISHIKAWA :—On the Formation of the Germinal Layers in Chelonia. *Quart. Jour. Micr. Sc. Vol. 27.* Also *Jour. Sc. Coll. Tōkyō, Vol. I. pt. 3.*

II. K. MITSUKURI :—On the Fœtal Membranes of Chelonia. *Jour. Sc. Coll. Tōkyō. Vol. IV. pt. 1.*

III. K. MITSUKURI :—Further Studies on the Formation of the Germinal Layers in Chelonia. *Jour. Sc. Coll. Vol. V. pt. 1.*

List of References.

No. 1. AGASSIZ, L. and CLARK :—Contributions to the Natural History of North America. Vol. II, Pt. 3. Embryology of the Turtle.

No. 2. DUVAL, M. :—Etudes histologiques et morphologiques sur les Annexes des Embryons d'Oiseau. *Jour. de l'Anat. et de la Physiol., XX. 1884.*

No. 3. DUVAL, M. :—De la Formation du Blastoderme dans l'Oeuf d'Oiseau. *Ann. d. Sci. nat. 6 ser. T. 18.*

No. 4. FLEMMING, W. :—Ueber Theilung und Kernformen bei Leucocyten und ueber deren Attractionssphären. *Arch. f. Mikro. Anat. Bd. 37.*

No. 5. HUBRECHT, A. A. W. :—The Development of the Germinal Layers of Sorex vulgaris. *Quart. Jour. Micros. Sci. Vol. XXXI.*

No. 6. KUPFFER, C. :—Die Gastrulation an der meroblastischen Eiern der Wirbelthiere und die Bedeutung des Primitivstreifs. *Arch. f. Anat. u. Phys., Anat. Abth. Jhrg. 1882.*

No. 7. KEIBEL, F. :—Zur Entwicklungsgeschichte der Chorda bei Säugern (Meerschweinchen u. Kaninchen) *Arch. f. Anat. u. Phys., Anat. Abth. 1889.*

No. 8. MEHNERT, E. :—Gastrulation und Keimblätterbildung der Emys lutaria taurica. *Morph. Arbeiten herausg. von Dr. Gustav Schwalbe, 3 Bd. 1 Heft. 1892.*

No. 9. RABL. C. :—Theorie des Mesoderms. *Morph. Jahrb., 15 Bd. 1889.*

No. 10. ROBINSON, A. and ASSHETON, R. :—The Formation and Fate of the Primitive Streak, with Observations on the Archenteron and Germinal Layers of Rana temporaria. *Quart. Jour. Micros. Sci. Vol. XXXII.*

No. 11. SARASIN, C. F. :—Reifung und Furchung der Reptilieneier. *Arb. a. d. Zool.—Zoot. Inst. zu Würzburg 6. Bd. 1883.*

No. 12. STRAHL, H. :—Die Dottersackswand und der Parablast der Eidechse. *Zeit. f. Wiss. Zool. Bd. 45.*

No. 13. VAN BENEDEN, E. :—Remarks made in the discussion following the reading of Rabl's paper on the Mesoderm. *Anat. Anz. 1888 p. 675*—Demonstration of the plates illustrating his investigations on the Formation of the Germinal Layers, the Chorda-Canal and Gastrulation in Mammalia. *Anat. Anz. 1888, pp. 709-714.*

No. 14. VIRCHOW, H. :—Das Dotterorgan der Wirbelthiere I. *Zeit. f. Wiss. Zool. 53 Bd. Supplement.* II. *Arch. f. Mikros. Anat. 40 Bd. 1 Heft. 1892.*

No. 15. WENCKEBACH, K. F. :—Der Gastrulationsprozess bei Lacerta agilis. *Anat. Anz. 1891 Nos. 2. u. 3.*

No. 16. WELDON, W. F. R. :—Note on the Early Development of Lacerta muralis. *Quart. Jour. Micros. Sci. Vol. XXIII. 1883.*

No. 17. WHITMAN, C. O. :—A Rare Form of the Blastoderm of the Chick and its Bearings on the Question of the Formation of the Vertebrate Embryo. *Quart. Jour. Micros. Sci. Vol. XXIII 1883.*

No. 18. WILL, L. :—Entwicklungsgeschichte der Gecko. *Biol. Centralbl. X Bd. Nos. 19 & 20, 1890.*

No. 19. WILL, L. :—Zur Kenntniss der Schildkröten-Gastrula. *Biol. Centralbl. XII Bd., No. 6, 1892.*

No. 20. ZIEGLER, H. E. :—Die biologische Bedeutung der amitotischen (directen) Kerntheilung in Thierreich. *Biol. Centralbl. II Bd., No. 12 & 13.*

No. 21. WILL, L. :—Beiträge zur Entwicklungsgeschichte der Reptilien. Die Anlage der Keimblätter beim Gecko (Platydactylus facetanus, Schreib.) *Zoöl. Jahrbücher, Abth. f. Anat. u. Ont., VI. Bd. I Heft.*

PLATE VI.

Plate VI.

FIG. 1.—Dorsal view of an embryo of *Chelonia caouana* a few hours after its deposition. Zeiss aa×2 (B1a)

FIG. 1*a*.—Ventral view of the same. aa×2

FIG. 2.—Dorsal veiw of an embryo of *Chelonia caouana* $1\frac{1}{2}$ days after its deposition. The lateral parts of the embryonic shield are not represented. Zeiss aa×2 (D1a)

FIG. 2*a*.—Ventral view of the same. aa×2

FIG. 3.—Dorsal view of an embryo of *Chelonia caouana* $1\frac{1}{2}$ days after its deposition. Only a small part around the primitive plate is represented. aa×2 (C2)

FIG. 3*a*.—Ventral view of the same. aa×2

FIG. 3. *bis*.—Dorsal view of an embryo of *Chelonia caouana* about 2 days old. Only a small part around the primitive plate is represented. aa×2 (O?)

FIG. 3*a*, *bis*.—Ventral view of the same. aa×2

FIG. 4.—Dorsal view of an embryo of *Chelonia caouana* $2\frac{1}{2}$ days after its deposition. Only a small part around the primitive plate is represented. aa=2. (B4a)

FIG. 4*a*.—Ventral view of the same. aa×2.

Fig. 4

Fig. 1

Fig. 2

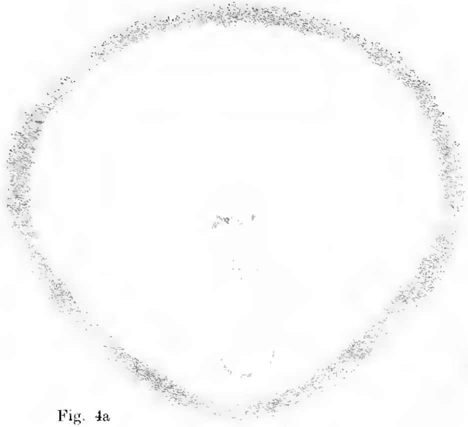


Fig. 4a

Fig. 3a bis.

Fig. 1a

Fig. 2a

Fig. 3a

Fig. 3

Fig. 3 bis.

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PLATE VII.

Plate VII.

FIG. 5.—Dorsal view of an embryo of *Chelonia caouana* $3\frac{1}{2}$ days after its deposition. aa × 2 (B5a)

FIG. 5*a*.—Ventral view of the same. aa × 2

FIG. 6.—Dorsal view of an embryo of *Chelonia caouana* $5\frac{1}{2}$ days after its deposition. aa × 2 (B7a)

FIG. 6*a*.—Ventral view of the same. aa × 2

FIG. 7.—Dorsal view of an embryo of *Chelonia caouana* $5\frac{1}{2}$ days after its deposition. aa × 2 (B7b)

FIG. 7*a*.—Ventral view of the same. aa × 2

FIG. 8.—Dorsal view of an embryo of *Chelonia caouana* $7\frac{1}{2}$ days after its deposition. aa × 2 (B9a)

FIG. 8*a*.—Ventral view of the same. aa × 2.

Fig. 6

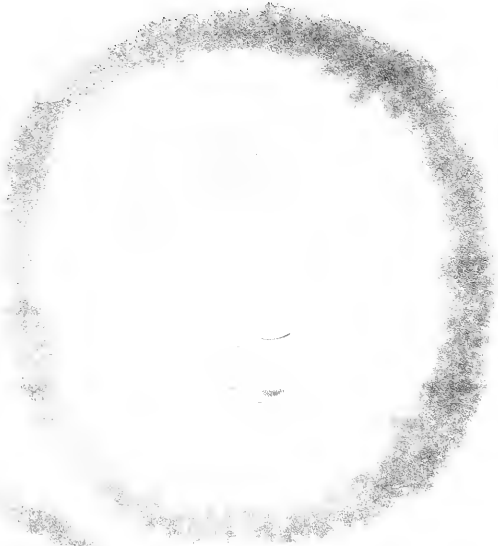

Fig. 5

Fig. 7

Fig. 8

Fig. 5a

Fig. 6a

Fig. 7a

Fig. 8a

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PLATE VIII.

Plate VIII.

FIG. 9.—Longitudinal section near the median line of an embryo of the same lot and stage as that represented in Figs. 1 & 1a. CC×2 (B1b, 4l. 2c. last—2)

FIG. 10.—Part of a longitudinal section of the same series as Fig. 9. From more lateral parts. DD×4 (B1b, long. 2, 5l., last—2)

FIG. 11 *a* & *b*.—Two merocytes from the same series as Fig. 9. DD×5 (B1b. {a. long. 2, 5l. last—2; b. long. 2, 5l. 7.s)

FIG. 12 *a* & *b*.—Cells of the same kind as those represented in the middle stratum of Fig. 10. DD×4 (B1b. {a. long. 2, 5l. last—2; b. long. 2, 3l. 6s.)

FIG. 13.—Longitudinal section near the median line of the embryo represented in Figs. 2 & 2a. CC×2 (D1a, 3l. 2c. 5s.)

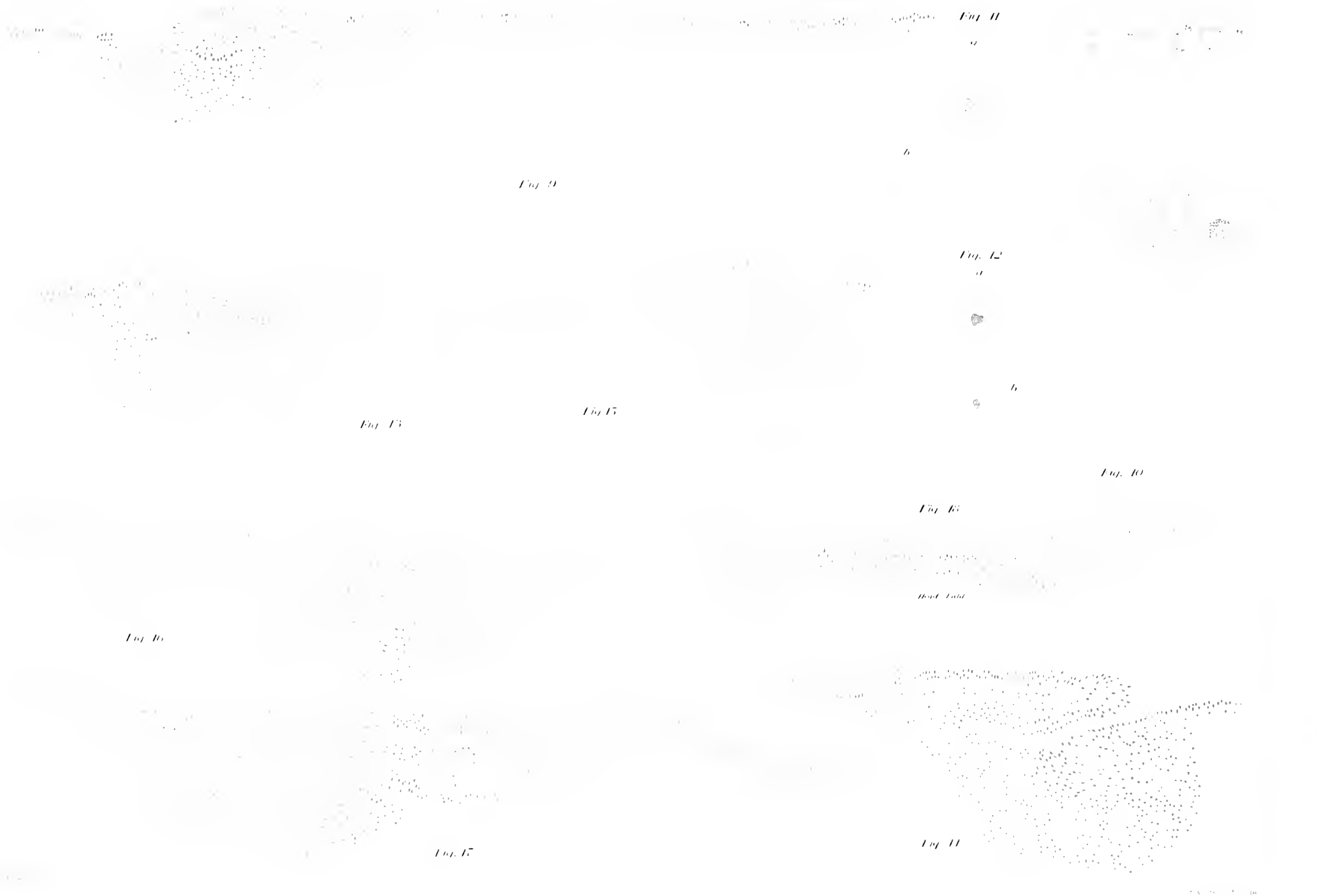
FIG. 14.—Longitudinal section near the median line of the embryo represented in Figs. 3 & 3a. CC×2 (C2. long. 1, 1l. 2c. 7s.)

FIG. 15.—Longitudinal section near the median line of an embryo 1 day older than Figs. 1 & 1a and 1 day younger than Fig. 4 & 4a. CC×2 (B3. long. 1. 4l. last)

FIG. 16.—Longitudinal section near the median line of the embryo represented in Fig 3 *bis*. and 3*a*. *bis*. CC×2 (O? long. 1. 2l. 2c. 2s.)

FIG. 17.—Longitudinal section near the median line of an embryo slightly younger than Figs. 6 & 6*a*. CC×2 (B6. 3l. 10s.)

FIG. 18.—Longitudinal section near the median line of the embryo represented in Figs. 7 and 7*b*. BB×2 (B7b. 3l. 2c. 1s.)



Note on the Eyes of Cardium Muticum *Reeve.*

by

K. Kishinouye, *Rigakushi.*

Zoologist to the Department of Agriculture and Commerce.

With Plate IX.

The eyes of the genus *Cardium* have been studied by many authors; but, so far as I know, no one has yet found such conspicuous eyes as I describe below in the molluscs of this genus. The following note may therefore prove interesting to investigators of the molluscan eye.

The mantle-edges, both right and left, of *Cardium muticum* are beset with dark brown, almost black pigment. They unite at the posterior end of the shell and form a triangular pigmented area, surrounding the siphonal openings. Over this area the right and left valves of the shell do not meet closely but leave a rather wide slit. In this triangular, pigmented area we find a great many tentacles, arranged in many irregular rows round the siphonal openings.

The tentacles are of various sizes: those on the periphery of the pigmented area are larger and longer; those towards the siphonal openings, shorter and more slender till at the margin of these they are reduced to mere fringes (fig. 1). Generally speaking the larger and longer tentacles about 100 in number bear the eyes. They are bent away from the siphonal openings, and each of them has a longitudinal band of black pigment on the siphonal side, *i. e.* on the

side exposed to the light.

The eye can easily be found without a microscope as a black spot on the siphonal side of the tip of a tentacle, opposite to the position of the eye of *Cardium edule*, which is "on the shell side of the mantle." * Along the upper, posterior † side of the eye, the epithelium is raised into a triangular screen (figs. 2, 6, 7, 9, *s.*). At one time I thought that this screen is the tip of the tentacle, but a closer examination showed that it is raised secondarily, probably to protect the eye, and that the true tip is occupied by the eye.

In the eye of *Cardium muticum* we find all the essential parts of an eye. Its structure resembles on the whole that of the eyes of *Pecten*, *Spondylus*, and *Cardium edule*; but differs in some essential points, since it is difficult to find the homology of some constituent parts.

The *cornea* (fig. 2, *c.*) consists of thin pavement cells, while that of other molluscs consists generally of columnar cells. The general epithelium becomes gradually thin as it approaches the lens and forms the thin cornea over it. The corneal cells are colourless, transparent, and polygonal in outline. The convexity of the cornea is great, its external surface being almost hemispherical.

The *lens* (fig. 2, *l.*) is large and consists of a great number of cells. Instead of the usual biconvex, flat form the lens of *Cardium muticum* is ovoid. Its longer axis is parallel to the optic axis and its broader end is directed below. It is more or less constricted at the middle part. The cells composing the lens are large, colourless and compressed in the direction of the optic axis, the degree of compression being greater nearer the cornea. In the median lon-

* Patten—Eyes of Molluscs and Arthropods. Mittheil. aus der Zool. Station zu Neapel. 1886.

† For the sake of convenience, the word *upper* is used to designate the distal end of the tentacle, and the word *anterior* that side of the tentacle turned towards the siphonal openings. The words *lower* and *posterior* naturally indicate the opposites of the above words.

gitudinal section, we find that the cells of the lens are arranged for the most part in two longitudinal columns. The cells are nucleated, having nuclei near the external surface of the lens.

The *retina* (fig. 2, *ret.*) is in contact with, and directly below, the lens. It is very simple in structure, consisting of colourless, columnar cells arranged in one layer. These cells have rods (fig. 2, *ro.*) which are not clearly found in the eye of *Cardium edule* (Patten,* Bütschli **). The rods are directed away from the retinophorae and are separated from them by a pseudomembrane. They seem to be hollow, as they appear as rings in cross section of some well preserved specimens. They are longest at the optic axis and gradually diminish in length as the distance from the optic axis increases. They are stained homogeneously and rather deeply. As regards the retinophoræ, the nuclei are deeply stained, while the protoplasm is very faintly stained. I could not find the ganglionic-cell layer, although it is stated to be present in the eyes of *Pecten* and *Cardium edule* (Patten, Bütschli).

Below the retina and continuous with it at the circumference, there is a layer of flat cells (fig. 2, *ch.*). These cells, are small and thin at the juncture with the retina, so that it is difficult to find out this connection in well developed eyes. I shall name this layer as the *choroid*, as it seems to be homologous in position to the pigment epithelium of the choroid of the vertebrates. If the tapetum (argentea of Patten) of *Pecten* and *Cardium edule* is cellular in orign, as it is stated to be, the choroid of *Cardium muticum* is probably homologous with it.

Below the choroid there is a membranous layer. It is the *tapetum* (fig. 2, *t.*). It consists of many thin, shining layers, stained deeply and homogeneously by colouring solutions. I cannot find any sign of

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* Patten—*loc. cit.*

** Bütschli—Notiz zur Morphologie des Auges der Muscheln. (Zoologisches Jahresbericht für 1886).

cellular structure, though according to Patten the like-named part of *Pecten* and *Cardium edule* is said to have been produced from cell layers. The tapetum covers entirely the lower surface of the choroid, except at the spots through which the branches of the optic nerve enter. If the tapetum of *Pecten* and *Cardium edule* is cellular in origin, there is in these *molluscs* no organ homologous with the tapetum of *Cardium muticum.* Moreover, the tapetum of *Pecten* and *Cardium edule* is said to be found below the layer of retinophoræ, while that of *Cardium muticum* is below the choroid.

Lastly there comes a *layer of pigmented cells* (fig. 2, *p.*). This invests all the external surface of the above described parts of the eye below the cornea, leaving only a small round area over the upper hemispherical portion of the lens or the pupil only. The pigmented layer as a whole is cup-shaped or rather urn-shaped. It consists of flat, polygonal cells arranged in one layer. The cells at the neck of the urn are smaller and thicker than those elsewhere. The pigment is black and serves to absorb rays of light which fall obliquely upon the retina. In *Pecten* the greater part of this function is fulfilled by the iris, and its red pigment layer, probably homologous with the pigment layer of *Cardium muticum*, absorbs rays of light from the lower side alone. As the eyes of *Cardium muticum* are destitute of the iris, the pigment layer is well developed.

The eyes are *innervated* from the viscero-parietal ganglia. A nerve (fig. 2, *n.*) runs through the central axis of each tentacle. It is divided into two branches where it touches the eye. One branch (fig. 2, n^2) passes through the pigment layer near the optic axis and spreads between the choroid and the tapetum, while the other branch (fig. 2, n^1) passes through the pigment layer at the level of the retina from the shell-side and seems to innervate the retina.

As the numerous eyes of an adult individual are sometimes found

in different stages of *development*, we can study their formation from their sections. It seems to me that the eyes of *Cardium* develop in two ways. Although we must regard both as more or less abbreviated, yet we can distinguish one as *the more abbreviated process* from the other *the less abbreviated*. If an investigation into their development could be made in immature specimens, more satisfactory results would doubtless be obtained, unless they are as in the young of *Pecten* in which Patten was disappointed to find any abbreviated development of the eye. I shall describe *the less abbreviated process* of development first.

There first forms on the tip of a tentacle (fig. 3) an invagination the mouth of which then closes (fig. 4). Thus a solid mass of cells is produced, with a slight concavity at the place where the mouth of the invagination at first opened, and this is turned towards the siphonal opening. It consists of cells in many irregular rows, while the general epithelium consists of columnar cells in one row. A nerve running along the central axis of the tentacle touches the lower surface of the cell mass and there divides anteroposteriorly into two branches.

In the upper part of the cell mass, a spherical portion becomes differentiated and separated from the rest by a basement membrane (fig. 5). This spherical portion is the rudiment of the lens, which is therefore epiblastic in origin, not mesoblastic as in the case of the eye of *Pecten*, according to Patten. The cells constituting the rudimentary lens are large and some of them have nucleoli. At this stage, although the lens is differentiated and separate from the surrounding cell mass, it is still imbedded in the epiblastic thickening.

In the next stage (fig. 6) the epiblastic thickening surrounding the lens is observed to be separated into two parts—an upper and a lower. The upper part is continuous with the general epithelium and consists of flat cells in one row. It is the cornea. The lower part is cup-shaped and is not connected with the general epithelium. It consists

of cells in about two rows. It is the retina. On the upper side of the base of the eye there is formed a large hollow. This makes the eye stand out and at the same time pushes it towards the siphonal opening. Thus the hollow divides the tip of a tentacle into two—a prominent eye and a somewhat triangular screen behind it. The hollow corresponds in position with the original mouth of the invagination for the eye. Pigment is produced in mesoblast cells surrounding the retinal portion. These pigmented cells form the pigment layer. A lumen is secondarily produced between the lens and the retina.

The lens grows by the multiplication of cells in the direction of the optic axis and assumes the shape of an ellipsoid, and consequently the lumen between the lens and the retina disappears (fig. 7). The peculiar arrangement of cells in the lens begins in this stage.

Late in development, the retinal portion is divided into two layers, the retina proper and the choroid. These two layers are continuous with each other at the circumference. Soon after the separation of the choroid from the retina, the tapetum is formed below the choroid, probably by the secretion of the cells which constitute the latter. I cannot corroborate the view that the tapetum is formed of modified cell layers, for even in these early stages I cannot find any thing of a cellular nature in it. Afterwards rods are produced from the retinal cells.

The more abbreviated process of development (see p. 283) is as follows :

At the top of a tentacle the epithelium becomes thickened and forms a little knob within (fig. 8). The little knob is next cut off from the epithelium (fig. 9). In this stage a hollow is produced behind the little knob and thus the triangular screen (fig. 9. *s.*) is formed. The little knob cut off is spherical in form and consists of a few, large

cells. It enlarges by the division of the cells and assumes an ellipsoidal form (fig. 10). Cells forming the lower part of the ellipsoidal mass become small by division. These smaller cells are not clearly distinguished from larger cells, as there are many cells of intermediate size. Later, however, the smaller cells are separated from the larger ones and form the retinal portion, while the larger become the lens. The later stages of development are quite like those in the first process.

In the two process of development, both the lens and the retina are produced from the epiblast. The process of their formation is probably abbreviated. Originally they were perhaps produced by two separate invaginations as in the case of the Vertebrata, one invagination for the retina and one for the lens. The invagination for the retina must have been the first to be closed and cut off from the epithelium; it formed a hollow sphere, the upper wall of which became the retina and the lower wall the choroid. The invagination for the lens was next formed and cut off in its turn from the epithelium, and had its lumen obliterated.

The eyes of *Cardium* differ from those of *Pecten** chiefly by the presence of the choroid between the retina and the tapetum and by the mesoblastic origin of the pigment layer; but I am inclined to think that in *Pecten* the choroid disappeared after secreting the tapetum, and that the red pigment layer is mesoblastic in origin, and not directly in connection with the retina.

* It is hard for me to accept Patten's observation that the lens of *Pecten* is mesoblastic in origin.

PLATE IX.

Explanation of Figures.

List of Abbreviations.

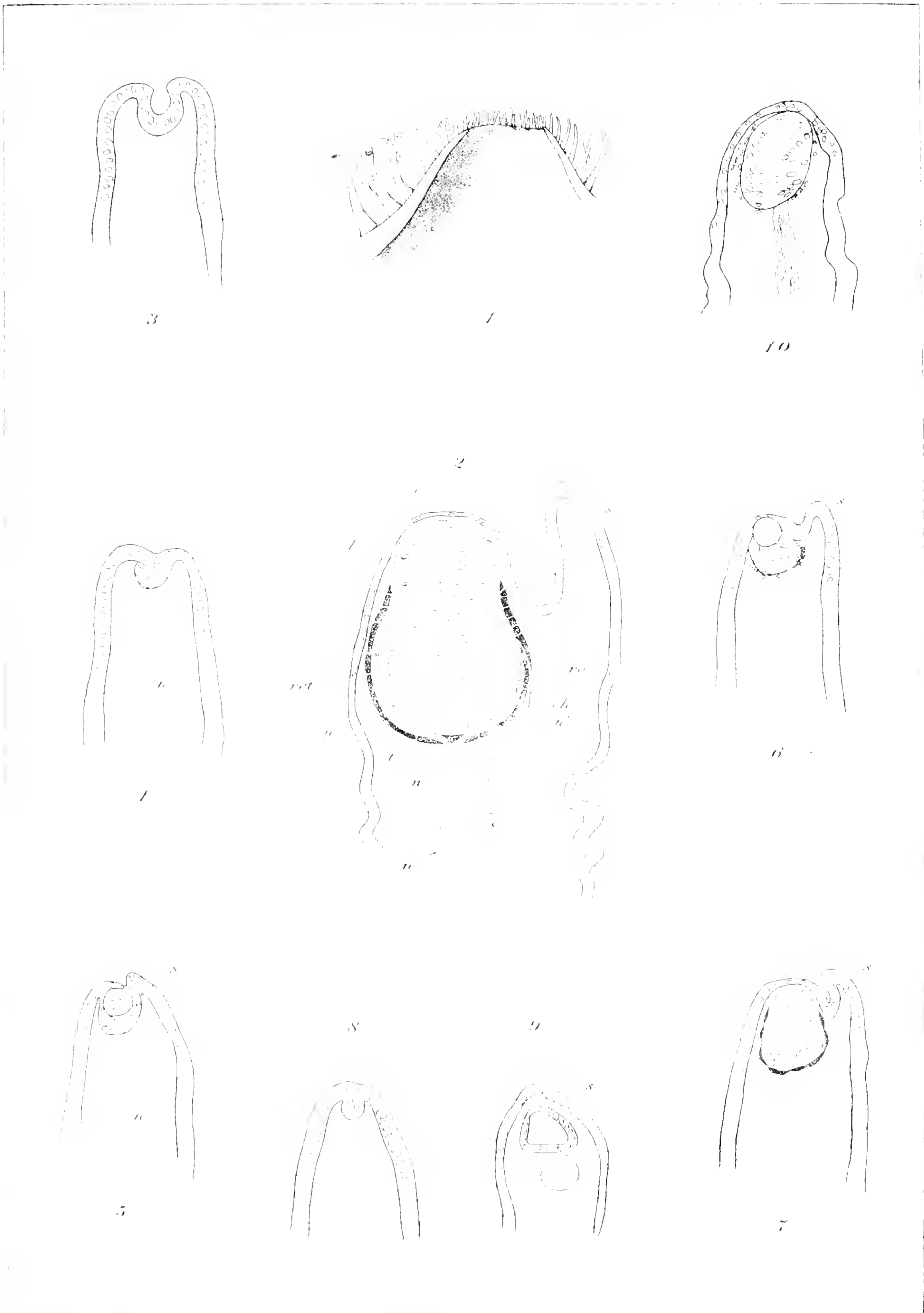
c.	Cornea.	ret.	Retina.
ch.	Choroid.	ro.	Rod.
l.	Lens.	s.	Triangular screen.
n, n^1, n^2.	Optic nerve and its branches.	t.	Tapetum
p.	Pigment layer.		

FIG. 1. Inhalent siphon of *Cardium muticum*, magnified. One longitudinal half of it is cut away.

FIG. 2. Median sagittal section of a fully developed eye (constructed from two sections).

FIGS. 3-7 Semi-diagrammatic representations to illustrate the less abbreviated process of the development of the eye (sagittal sections).

FIGS. 8-10. Semi-diagrammatic representations to illustrate the more abbreviated process of the development of the eye (longitudinal sections).



K. Kishinouye del.

Note on the Cœlomic Cavity of the Spider.

by

K. Kishinouye, *Rigakushi.*

Zoologist to the Department of Agriculture and Commerce.

With Plate X.

The species investigated belong to the genera, *Lycosa* and *Agalena*.

My object in undertaking this investigation was to ascertain whether the stercoral pocket is mesodermic in origin as I described in my former paper* or ectodermic and a part of the proctodæum, as is generally believed.

When the embryo has reached the stage in which there are nine segments in the ventral plate, *i.e.*, seven segments between the cephalic and caudal lobes, the mesoderm is divided into as many segments, while a pair of the cœlomic cavities appears almost simultaneously in each of the first five segments, exclusive of the cephalic lobe (Fig. 1).** As at this stage the segment of the cheliceræ is not yet formed, the cœlomic cavities now formed belong to the segments of the pedipalpi and the four ambulatory appendages. The rudiments of appen-

* On the Development of Araneina. This Journal. vol. IV.

** As the segmentation of the ventral plate seems to take place simultaneously with the division of the mesoderm into somites, I will hereafter for the sake of brevity, simply describe the formation of a segment and will not mention every time the formation of the corresponding mesodermic somite.

dages make their appearance in these five segments as small round knobs. The cœlomic cavities of the same segments are just below the evaginations for the appendages.

In the next stage the segment of the chelicerae is cut off from the cephalic lobe, and three new segments are added between the last segment and the caudal lobe (Fig. 2). By this time the cephalothoracic appendages have become much elongated, and the mesoderm has spread into them. The cœlomic cavity extends to the distal end of each appendage ; in fact the larger part of the cavity is now found within the appendage. It spreads a little at the base of the appendage, so that three portions may be distinguished in it in a cross section of the embryo (Fig. 5) as Schimkewitsch* observed—one portion in the appendage, a second extending a little in the dorsal direction, and the third extending a little ventrally. The last two are the horns of the basal enlargement. In the abdominal region, the provisional appendages are not yet formed ; but the mesodermic somites develop rapidly, and in each of the first and the second abdominal segments a pair of cœlomic cavities is produced. Thus seven pairs of the cœlomic cavities are now found.

Subsequently two more new segments are added between the last formed segment and the caudal lobe. A pair of the cœlomic cavities is formed in this stage in each of the following segments: the cephalic lobe, the segment of the chelicerae, and the third to the seventh abdominal segments (Fig. 4). These seven newly formed pairs of cœlomic cavities together with the seven pairs already existing make in all fourteen pairs, the caudal lobe alone being now devoid of any. In the cephalic lobe the mesoderm is not divided into two lateral parts, therefore the two cœlomic cavities, right and left, are separated

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* Étude sur le Développement des Araignées. Arch. de Biologi . Tome XI. 1887.

by the median partition of mesodermic cells; while the right and left cœlomic cavities of the other segments are separated by the yolk (Figs. 3, 5).

The cephalothoracic appendages, with the cœlomic cavities in them, elongate very much, and bend towards the ventral median line (Fig. 5). The mesodermic somite and also the cœlomic cavities of the first abdominal segment have developed little since their formation. Korschelt and Heider state in their text book that in this segment a pair of the provisional appendages is formed as in the four succeeding segments. Though a pair of slight elevations is found in this segment, they are very much lower than the provisional appendages of the succeeding segments, and moreover they are chiefly of the ectodermic thickening (Fig. 4). I am therefore inclined not to call them provisional appendages. In each of the second to fifth abdominal segments, a pair of provisional appendages appears. They are short elevations of the ectoderm, into which the cœlomic cavity enters, as in the case of the cephalothoracic appendages. The cœlomic cavities of these segments develop in the dorsal direction: hence in cross sections of the embryo these abdominal somites differ from the thoracic segments in having a shorter branch of the cœlom in the appendage, and a longer one towards the dorsal side (Fig. 5). The ectodermic cells covering the mesoderm somites are always high and columnar in shape, and are easily distinguished from the cells of other parts.

Soon afterwards one more new segment is cut off from the caudal lobe, and in its mesodermic moieties a pair of cœlomic cavities appears. At this stage, the ventral plate attains the greatest antero-posterior extension round the egg, so that the cephalic and the caudal lobes almost touch each other (Figs. 3, 6). The caudal lobe is raised a little above the general surface of the egg. I stated erroneously in my first paper that the mesoderm of the caudal lobe is split to form

an unpaired cavity in this stage. In the abdomen the growth of the mesodermic somites, except that of the first abdominal segment, is enormous, extending rapidly towards the dorsal median line. Thus in the abdomen, the dorsal portion of the cœlomic cavities develops rapidly, while their ventral portion as well as the portion which enters into the appendage remains only slightly developed. In the cephalothorax, on the contrary, the portion of the cœlomic cavities which enters into the appendage develops rapidly, while their ventral and dorsal portions remain undeveloped.

The reversion of the embryo now begins, and when the process advances a little, the two nerve cords and the appendages of both sides of the ventral plate, or the two lateral divisions of the germinal band, begin to separate from each other. This lateral extension of the ventral plate, together with the rapid growth of the dorsal portion of the mesodermic somites in the abdomen, causes the dorsum of the embryo to elongate longitudinally. At this stage the mesoderm of the cephalothorax shows no noteworthy changes. The cœlomic cavities of the cephalic lobe develop anteriorly, or towards the dorsal region (Figs. 7, 8). The first abdominal segment begins to degenerate; its mesodermic moieties and ganglia may however be seen with some difficulty. In the second to fifth abdominal segments the mesodermic moieties develop greatly towards the dorsal median line, so that they nearly meet each other at that line (Fig. 7). The ectoderm covering these mesodermic somites is elevated a little, and forms the so-called tergal portion of the abdominal segment. Locy* illustrates his paper with a figure (Pl. II. Fig. 9) in which a pair of terga may be seen before the second abdominal segment; but he gives no description of them. I myself am unable to find them. He says

* On the Development of Agalena nævia. Bull. Mus. Comp. Zool., vol. XII. 1886.

also that at first "the only dorsal elements developed were the five pairs belonging to the abdominal somites (second to sixth), but during this stage the dorsal elements of the limb-bearing somites begin a more rapid growth." But according to my observation this is not the case, the dorsal elements of the limb-bearing somites remain undeveloped as in the previous stage.

Between two consecutive terga there is a furrow. This furrow was described by Schimkewitsch* as having no relation with the mesodermic somites. He says that it is due to the mesodermic somites fusing together, before they develop dorsally to cover the dorsal surface. He says moreover that the number of these furrows never corresponds with that of the somites. My observations as given above do not corroborate these statements.

The last three abdominal segments (sixth to eighth) gradually degenerate and their cœlomic cavities seem to fuse together into one pair. The pair of the cœlomic cavities thus formed by fusion is pushed into the protuberance of the tail as the process of reversion proceeds (Fig. 7). I find that fusion of the cœlomic cavities does not take place before this stage though Schimkewitsch says that it does in the cephalothorax and in the abdomen and I also erroneously stated that it occurs in the thorax before this stage.

A cavity is produced in the mesoderm of the tail lobe. It is unpaired (Figs. 7, 8). The unpaired cavity thus made cannot be conceived otherwise than as a homologue of the cœlomic cavity. Though the cavity is certainly not formed by an invagination, I thought that the cells in the tail lobe might be produced by the proliferation of the ectoderm. But I found that the cells enclosing the unpaired cavity are the remnant of the mesoderm cells which gave rise to the meso-

* loc. cit.

dermic somites of many preceding segments and that they are entirely separate from the ectoderm. Previous authors who have studied the development of the spider, overlooked this cavity in the mesoderm of the caudal lobe, and observing the stage at which the unpaired cavity communicates with the proctodæum consider the former as a portion of the latter. If the cells enclosing the unpaired cavity are ectodermic in origin, the numerous mesoderm cells in the caudal lobe must disappear all at once, as there are no cells in the lobe except those surrounding the last fused pair of cœlomic cavities. But the disappearance of many cells at once is quite impossible.

As the process of reversion proceeds still further, each half of the pair of the cœlomic cavities in the cephalic lobe is divided into two portions—one at the side of the stomodæum (Figs. 9, 10, *1 cœl. b*), the other below the anterior border of the semicircular groove of the brain (Figs. 9, 10, *1 cœl. a*). The former disappears soon afterwards but the latter elongates towards the dorsal median line (Fig. 11), and the mesodermic walls of the cavities of two sides meeting at the median line fuse together, leaving, however, a canal between them. This canal is the aorta.

The portion of the egg which is not covered by the ventral plate or the abdominal terga is characterised by the absence of the mesoderm cells and by the presence of the secondary endoderm cells or the fat cells instead. The secondary endoderm cells are directly under the ectoderm. Most of them are enclosed between the walls of the mesodermic moieties of the abdominal somites and become the blood corpuscles.

The cœlomic cavities in the segments of the cheliceræ and the pedipalpi degenerate and disappear. The greater portion of the cœlomic cavities of the four ambulatory appendages degenerates, the mesoderm cells forming their wall becoming gradually changed into muscles.

Their proximal and outer portions remain distinct at the base of each leg (Fig. 9). The cœlomic cavity of the first ambulatory appendage communicates with the exterior by means of a duct which is produced by an ectodermic invagination. The first abdominal segment disappears entirely.

The mesodermic moieties of the second to fifth abdominal segments, or the segments bearing the provisional appendages, meet one another at the dorsal median line and form there the wall of the heart. The wall of the cœlomic cavity of the second abdominal segment meets that of the cephalic lobe (Figs. 9, 10). The formation of the dorsal circulatory system, in which some thoracic somites do not take part, resembles greatly that of Limulus.* As a lateral slit or ostium is made where two consecutive somites meet, the number of the slits in the adult heart shows approximately the number of the segments which took part in the formation of the heart.

The sixth to eighth abdominal somites are entirely degenerated, their mesoderm cells are disintegrated and fill the caudal lobe at the sides of its unpaired cœlomic cavity (Figs. 9, 10). The latter becomes wide, and over its posterior end, the ectoderm is slightly invaginated (Fig. 9). The invagination is the rudiment of the proctodæum, so that the unpaired cavity of the caudal lobe is produced independently of the proctodæum.

In the next stage in which the embryo assumes the ventral flexure and the constriction between the cephalothorax and the abdomen appears, the cœlomic cavities undergo great changes. In the cephalothorax, they all disappear, except the small portions at the outer bases of the first to third ambulatory legs. These remnants fuse together and form the coxal gland (Fig. 12). The lumen or the

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* Kishinouye—On the Development of Limulus longispina. This Journal, vol. V.

cœlomic cavity of the gland is so small at this stage that the gland seems almost solid. In the abdomen also, all the cœlomic cavities disappear except the unpaired one in the caudal lobe, which, inexplicable as it may seem, remains as the stercoral pocket, as was described and figured in my former paper. At this stage the mesoderm cells at the intersegmental portions grow inward into the yolk and form the dissepiments, these, being specially well developed between the second and the third, the third and the fourth, the fifth and the sixth abdominal segments (Fig. 12).

Those portions of the embryo which were destitute of the mesoderm in the stage of Fig. 9 now dwindle, according as they are encroached on either by the ventral plate or by the abdominal terga in their growth: in the cephalothorax, the cephalic lobe is bent towards the dorsum and its lateral margins fuse with those of the ventral plate of the thorax, while in the ventral portion of both the cephalothorax and the abdomen, the two lateral cords of the nervous system with the underlying mesoderm meet each other at the ventral median line.

PLATE X.

Explanation of Figures.

Reference Letters.

abd. seg.	abdominal segment.	*pedip.*	pedipalpi.
br.	brain.	*proct.*	proctodæum.
ch.	chelicerae.	*sem. gr.*	semicircular groove.
c. l.	caudal lobe.	*st. p.*	stercoral pocket.
cœl.	cœlomic cavities.	*stom.*	stomodæum.
cox. gl.	coxal gland.		

FIG. 1. Side view of an embryo at the stage in which the first rudiments of appendages have appeared.

FIG. 2. Side view of an embryo at the stage in which all the cephalothoracic appendages have appeared.

FIG. 3. Dorsal view of an embryo at the stage of the maximum ventral flexure.

FIG. 4. Sagittal section of an embryo at the stage of Fig. 3.

FIG. 5. Cross section of an embryo at the stage of Fig. 3.

FIG. 6. A portion of the median sagittal section of an embryo at the stage of Fig. 3.

FIG. 7. Oblique side view of an embryo at the stage of reversion.

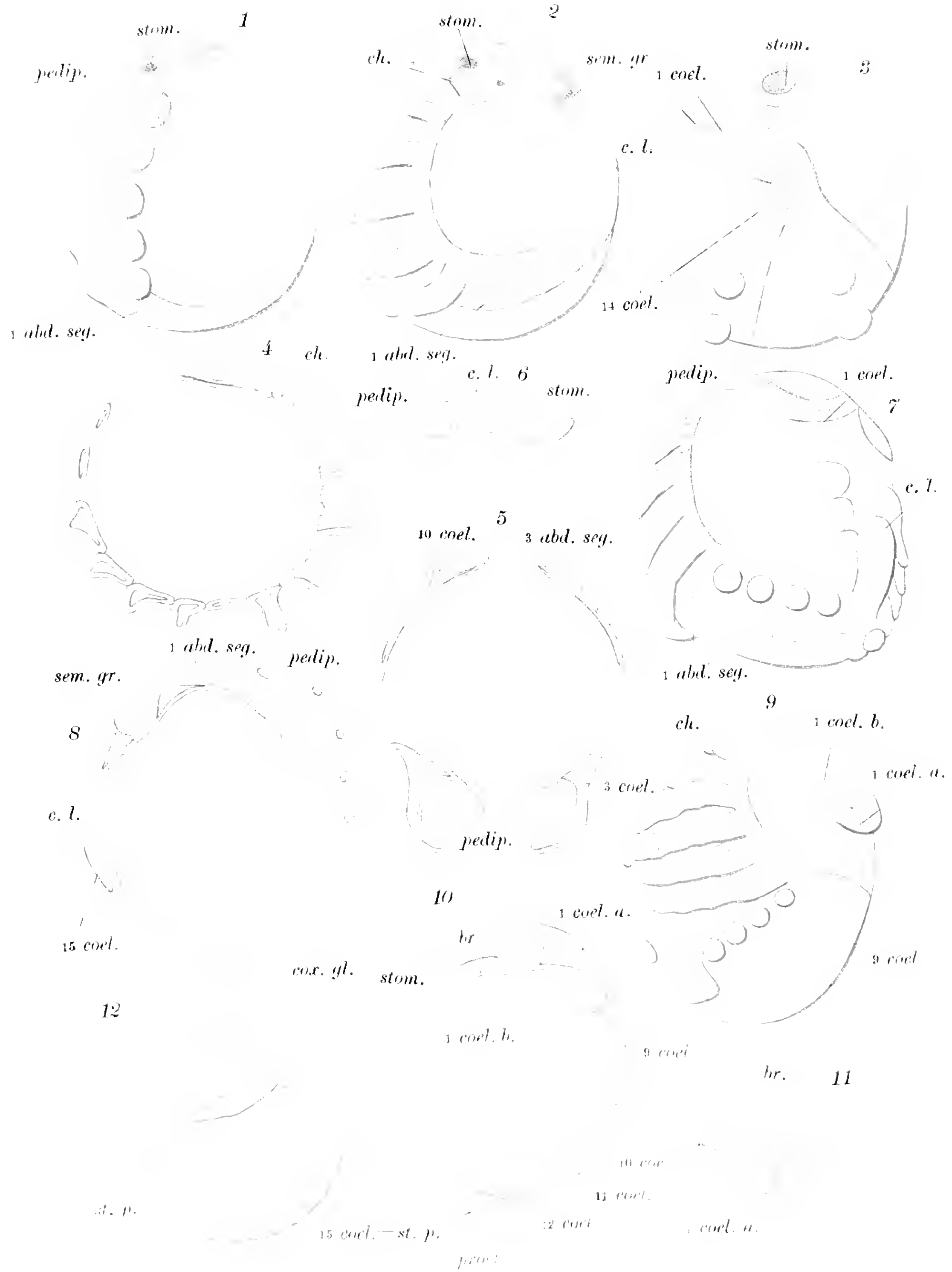
FIG. 8. Oblique sagittal section of an embryo at the stage of Fig. 7.

FIG. 9. Side view of an embryo at the stage in which the process of reversion has greatly advanced.

FIG. 10. Oblique sagittal (about median) section of an embryo at the stage of Fig. 9.

FIG. 11. Frontal section of the cephalic region of an embryo at the stage of Fig. 9.

FIG. 12. Side view of an embryo at the end of the process of reversion.



K. Kis... del.

... Imp. The Tokio-Seishibunsha

Studies of Reproductive Elements.

II. Noctiluca miliaris, Sur.; its Division and Spore-formation.*

By

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With Plates XI–XIV.

A. DESCRIPTIVE PART.

a. Division.

The individuals of the genus Noctiluca which are preparing to divide can easily be distinguished from the others by their spherical form, caused by the absence of the peristome and of teeth, and by the diminished size of the tentacle (Figs. 1 and 2); but the *Staborgan* is always to be seen as a narrow line of cytoplasm, extending backwards from the nucleus and marking the line of division. The tentacle appears to be drawn into the body of the animal, as *Cienkowsky* (**8**, p. 50) first observed, and not thrown off, as Robin (**6**, p. 1064) is

* The materials upon which the following observations were carried out were in part collected by myself from different parts of the sea coast near Tokyo, for instance, in Numadsu, Zushi, Tateyama-in-Boshiu, etc., and in part from the Misaki Marine Biological Station of the Science College of the University, where I worked for a fortnight during the summer of 1890, for a week during the winter of 1892-93 and again for a week at the end of August this year. For all these opportunities and for the free use of the library of the Zoological Laboratory of the Science College, I express my warmest thanks to the authorities of the College, and especially to Professor Mitsukuri.

inclined to think. My figures (1 and 2) show the stages of its withdrawal.

The central protoplasm, which in ordinary conditions takes a more or less elongated form, then becomes concentrated into a star-shaped mass around the nucleus. Close to the nucleus, part of it differentiates itself from the rest by coarser granulation. It is usually spherical in shape (Figs. 1 and 2), but sometimes elliptical (Fig. 31.). This cytoplasmic mass represents the kinetic centre of division, as will be later shown, and corresponds to the *sphère attractive* of *van Beneden*, or *Archoplasma* of *Boveri*. In many cases this archoplasmic mass is formed before the complete disappearance of the peristome, teeth, etc., as is shown in Fig. 31. The relative position of the archoplasm and the nucleus does not seem to be always the same; but both bodies lie in the direction of the *Staborgan*, as will be seen in Figs. 1, 2, and 31. In Figs. 1 and 31 the archoplasm lies behind the nucleus, whereas in Figs. 2 and 3 it lies in front of it. In living animals the nucleus at this stage appears as a transparent vesicle, and the archoplasm is very distinct, owing to coarse granulation.

The archoplasm now stretches itself in a direction more or less in the vertical plane passing through the *Staborgan*, and forms a relatively large spindle (Fig. 3.). It undoubtedly corresponds to the central spindle of *Hermann* (**20**, p. 580). This spindle gradually elongates and assumes a position longitudinal to the line of division, and divides the nucleus (Fig. 4.). The central mass becomes at the same time separated into two portions, each concentrating around its archoplasm. A narrow line of cytoplasm now appears also in front of the central mass in the direction of the *Staborgan*. The spindle parts at the middle, dividing the nucleus into two daughter nuclei. These (the daughter nuclei) at first lie in the line of division, but soon change their positions and come to lie right and left of it

(Figs. 5-9). Which of the two nuclei goes into the one half and which into the other, is not easy to ascertain, as a narrow line of cytoplasm, as above stated, now stretches also in front of the nucleus nearly in the same manner as does the *Staborgan* behind, so that the fore and hind ends are difficult to distinguish. But in all ascertained cases, the front nucleus goes into the right side and the hinder one into the left. All the while the archoplasm remains close to the nucleus. The line of division gradually extends both ways towards opposite surfaces of the body, and the animal assumes the well-known biscuit-shape. Meanwhile, the two protoplasmic masses with their respective nucleus and archoplasm shift their positions and come to lie on each side of the line of division, nearly opposite to each other (Fig. 9); so that division of the animal now appears to take place in the longitudinal plane. The relative position of the nucleus and the archoplasm does not seem to remain the same during the earlier stages of the division, (Figs. 5-9), but at later stages the former lies farther away from the dividing line (Figs. 10-12).

The time during which all these changes take place, from the stage shown in Fig. 3 to the stage of Fig. 9, varies greatly in different individuals; but from observations upon fourteen animals kept in a moist chamber at the Misaki station, during the latter part of August 1893, the time required varies from six to eight hours.

The division of the body proceeds, until only a small connecting bridge of cytoplasm is left (Figs. 10 and 11). During this stage, tentacles usually grow on both halves, which swim together like the Siamese twins. Such a couple, with rather short tentacles, is shown in Fig. 11. That shown in Fig. 12 has tentacles much more developed. This figure is drawn from a living specimen, so that the nuclei (*n*) appear quite transparent, while both the archoplasms (*a*) are seen as

dark granular masses close to the nuclei. The protoplasmic connection (*p. c*) is reduced to a narrow bridge.

The changes which take place in the nucleus as well as details of the nuclear division will be spoken of in connection with the process of spore-formation.

b. Spore-formation.

Spore-formation is always preceded by the concentration of the central plasm into a rather small area, and by the disappearance of all other structures, such as the tentacle, flagellum, teeth, etc. The individual appears, therefore, as about to divide, except in not possessing mouth and *Staborgan*. Its central plasm is also raised a little above the general surface of the body, and forms a low knob-like elevation, as has been observed both by *Cienkowsky* (**89**, p. 54) and, by *Robin* (**6**, p. 1069). Just as in division, we find here a spherical concentration of the granular cytoplasm about the nucleus, which is also quite transparent in living specimens. Thus, Fig. 13, which is drawn from an individual in this stage, shows the nucleus (*n*) as a spherical vesicle, while the archoplasm (*a*) appears as a granular mass. The division of the nucleus is also preceded here by that of the archoplasm, and we thus obtain a stage, represented in Fig. 14, which exactly corresponds to the stage of division represented by Fig. 4. In what direction the spindle lies in this case, it is not, however, possible to tell, as the external signs of the bilateral nature of the animal can not all be recognised. The division proceeds still further, till the stage is reached represented in Fig. 15 or 16, the latter of which is drawn from a living specimen. We sometimes meet with a sort of partial segmentation of the entire body at this stage, the plane of division cutting the connecting axis of the two nuclei, as has been observed by *Cienkowsky* (**7**, p. 134). Fig. 17 represents a stage with four nuclei, on the side of each of which is seen a large archoplasmic

spindle, showing thus their further division. The cytoplasm surrounding each of the nuclei and archoplasms is connected with the others by a fine network, like the protoplasm in the segmentation of Leptodora eggs observed by *Weismann* and by myself (**37**, Taf. II, Fig. 30). Fig. 18 gives a side view of a four-nucleated stage, in which the nucleus on the right side is already divided into two daughter nuclei, while that on the left is in the spindle form. Fig. 19 represents a stage immediately succeeding this. Here, in this individual, is seen a partial line of division of the entire body, marking its surface into two hemispheres. A further stage is represented by Fig. 20, where we see six nuclei preparing to divide. Of these two belong to one half and four to the other, showing thus that the division of the buds does not occur quite regularly even at this early stage. The first line of division, although now very faint, is still clearly to be recognised. In the living condition the body of the Noctiluca had been imperfectly divided into two portions at the part where the nuclei lie. This is shown more distinctly in Fig. 21, where ten nuclei are to be found. Of these six belong to one half and four belong to the other. The partial line of division of the body is also plainly to be seen in this specimen. Fig. 22 represents a further stage of budding, viewed laterally. All the nuclei visible are represented in this figure, and their number is fifty-one, showing again the irregularity of division. A further stage of sporulation is seen in Figs. 23-30. Figs. 23-26 are drawn from sections of nearly ripe spores, while those represented by Figs. 27-30 are from preparations of entire spores, drawn under different magnifying powers. In Figs. 25,26 (the spore on the right hand side of the figure), and 27-30, the spore cells are quite separated from each other, and they have all got a long flagellum springing out from the basal third of the cell, where the archoplasm lies.

c. Nuclear Division.

In a short paper[1] sent to the *Naturforschenden Gesellschaft zu Freiburg im Breisgau*, I treated at some length of the structure of the nucleus in the resting stage, as well as of the peculiar manner in which the division of the Noctiluca-nucleus takes place. As the present account covers the same ground, some of the facts described in that paper are necessarily repeated in this one. Nevertheless I do not hesitate to go again over the same subject here, for two reasons, namely: (*a*) many new facts have become known, and the nature of some of the doubtful phenomena been ascertained since then; (*b*) the papers appeal to two different sets of readers.

1. *Nucleus in resting stage:*—The living nucleus appears quite transparent and homogeneous, as all other observers of Noctiluca have long ago described. It is covered by a rather thick membrane, whose contour is often seen as double. When treated with reagents, such as acetic acid, osmic acid, chromic acid, etc., a number of chromatic bodies can easily be distinguished in it. These elements appear as strings of deeply staining bodies which often take an **S**-form. Each string consists, in well prepared specimens, of a number of disc-shaped microsomes arranged one after the other like the chains of mammalian red blood-corpuscles. These strings lie either singly or, more often, two or four united together. In the paper named above, I supposed the single string to be primary, and the compounds of two or four to be secondary, probably produced by division of the single string. In accordance with this supposition, I designated the number of chromosomes as ten. In many cases this number seems to hold true, as Figs. 33, 34, 40-47 show; but in other cases, such

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[1] This paper was written for the *Festschriften* commemorating Prof. Weismann's 60th birthday which occurs on the 17th January, 1894.

as Figs. 35 or 36, the number of chromosomes counts greatly above this, and I am not, at present, able to tell exactly their number.

2. *Prophasis*:—Figs. 31, 33, and 34 are taken from preparations of nuclei which are just preparing to divide. In Fig. 31 the chromosomes are not very distinctly to be seen, owing to the deep staining with hæmatoxylin. We see, however, on the left hand side a curved double line undoubtedly representing the segmentation of one of the chromosomes. This is very beautifully seen in preparations represented by Figs. 33 and 34, both of which are stained with a combination of acid-fuchsin and methylen-blue. In both nuclei the chromosomes are seen to be composed of a double row of minute microsomes, the number of which in a single chromosome is not with certainty to be determined, since the entire length of a chromosome is very rarely to be made out. In a few cases, however, I counted them to be ten or twelve in a single row, as will be seen in the **V**-shaped chromosome, lying a little to the left, in the nucleus represented by Fig. 33. A nearly equal number will be counted, though not so clearly, in one of the two chromosomes at the right hand of the same nucleus. Similar states of chromosomes are seen in Figs. 44-47, drawn from preparations of spore-forming individuals in about the same stage as that represented by Fig. 18. In Fig. 44 the chromosomes, except a few, do not seem to lie in any definite order, but to be scattered more or less irregularly in the nucleus; but in Figs. 33, 34 and 45, they lie more or less parallel with each other, their long axis being in the direction of the archoplasm found just external to the nucleus. In Fig. 46 the above position of the chromosomes is not so clearly expressed as in the figures just mentioned, although the archoplasm can be distinctly seen on one side of the nucleus. This, however, is only an optical illusion caused by the position of the archoplasm which does

not lie in the plane of the paper, but a little beneath it. By focussing the tube, however, we can distinctly observe the chromosomes as radiating from the pole where the archoplasm lies. A close examination of the chromosomes, represented by Figs. 44-47, of the spore-forming individuals, and those, Figs. 33 and 34, of the nuclei just before division, seems to show that there is some marked difference between the two. While the chromosomes of the nuclei of the dividing individuals are represented by a double row of microsomes, those of the nuclei of the spore-forming individuals appear to consist of four rows (comp. Figs. 47 and 33). This difference is undoubtedly due to the division and the spore-formation. *In division the nucleus has to divide only once, and hence the chromosomes require only once to divide, while in the spore-formation divisions of the nuclei take place rapidly one after the other, and two divisions take place almost simultaneously.* The way in which two divisions take place simultaneously can be made out from Figs. 44 and 47, where some of the chromosomes are seen at their ends. The microsomes become thicker at their periphery, and form a row of rings. Each chromatin-ring or, perhaps more correctly, the microsome-ring becomes thickened at four places and the interspaces between these thickenings break up and thus form four daughter microsomes. Sometimes the thickening does not take place simultaneously at four places and the ring does not break up into four at once, as will be seen in a ring at the right hand side of Fig. 47, where it is thickened only at one point. Another ring at the left hand side is broken at one place, and appears C-shaped. A ring represented on the right hand side of the nucleus Fig. 44, shows a very interesting stage. This ring consists of two thick curves connected together by narrow cementing substance—the *linin*-thread. The ends of these curves are again thickened, showing that the division of the ring into four daughter microsomes does

not in reality take place simultaneously, but by a succession of two divisions rapidly following one after the other.

3. *Spindle stage*:—In my paper for Prof. Weismann's birth-day celebration I have given a figure of a nucleus and an archoplasm in the latter of which could be observed a clear space next the nucleus. This space was in apparent communication with the interior of the nucleus. Since then I have been enabled to find a number of fibres in this space (Fig. 46), but the existence of any communication between these fibres and the interior of the nucleus can not with certainty be made out. The archoplasm becomes elongated and forms a very large spindle. This corresponds undoubtedly with the central spindle of *Hermann*, as stated above, but will be called the ARCHOPLASMIC SPINDLE, in distinction from other spindles already described in karyokinetic divisions of other animals and whose origin is still under discussion; while the fibres of this spindle will be called the CENTRAL FIBRES, for reasons stated below. At first the archoplasmic spindle lies tangentially on the surface of the nucleus, but soon assumes a curved figure, depressing at the same time the nuclear wall. In consequence of this the nucleus takes a broad C-shaped figure, in the concavity of which the archoplasmic spindle lies. A notion of these stages may be gathered from Figs. 3, 21, and 24. The chromosomes which, at the stages of Figs. 45, 46, 48, &c., are seen radiating from a single point, become with the division of the archoplasm separated into two masses, one attached to each end of the spindle. The archoplasmic spindle elongates and the nucleus assumes the form represented by Figs. 20, 35, 36, 49, and 52. Fig. 35, prepared from specimens killed with *Flemming's* fluid and stained with *Böhmer's* hæmatoxylin, represents the division of the nucleus in a dividing individual. The nucleus assumes more or less the shape of a dumbbell, and the archoplasms at both ends

of it are irregular in outline. The chromosomes have now distinctly divided into two portions radiating from the pole of the nucleus where the archoplasms lie. Each chromosome consisting of a number of microsomes arranged like beads, is bent upon itself, as is so usual in this stage of karyokinesis in other animals. The number of these chromosomes, as stated above, is very difficult to ascertain. The most interesting phenomenon in this figure (35) is the spindle fibres, of which there are two kinds; those, which I call CENTRAL FIBRES (Figs. 35, 36, 37, 48, *cf.*), lying imbedded in the concavity found in the wall of the nucleus, and standing in no direct relation with the nucleoplasm; and those (Figs. 35, 36, 37, 48, *rf.*) which are seen at the side of the central fibres, running from the centre of an archoplasm to the bent ends of the chromosomes, and which I will call the RADIATING FIBRES. Where they join the nuclear wall, the latter appears quite indistinct, but whether they pass through the wall of the nucleus at these points and are directly attached to the chromosomes can not be made out. In Fig. 37, which represents a section of a dividing nucleus at about the same stage as Figs. 35, 36, and 37, a homogeneous mass of nucleoplasm is found at the poles of the nucleus, external to the bent ends of the chromosomes. In this mass of nucleoplasm, which is easily distinguished from the more granular archoplasm lying outside, are seen many parallel fibres running in the same direction as the radiating fibres of the archoplasm represented in Figs. 35, 36, and 48, but unfortunately not to be seen in the series of sections represented by Fig. 37. We shall come to this again in the part devoted to general considerations, as further discussion of the subject is beyond the region of observation.

When the division has proceeded as far as Fig. 35 or 36, the median portion of the archoplasmic spindle is swollen up a little, as

will be seen in these two figures and also in Fig. 4. This part of the spindle is left behind after the complete separation of the nuclei in the form of a small diagonal figure (Fig. 39 *x*). What becomes of this figure is not ascertained. Fig. 35 is drawn from preparations of a specimen killed with picro-acetic acid and coloured with fuchsin-methylen blue. The archoplasm, and the central as well as the radiating fibres are seen as in Fig. 36, but some of the chromosomes are here more or less swollen up, owing perhaps to the action of the acetic acid. Fig. 35 *A*. is drawn from the same nucleus as that of Fig. 35 but at a level lower than that of the archoplasmic spindle. It should be mentioned by the way that the spindle lies always nearer the surface of the cell than to the nucleus. In both these figures will be seen a number of narrow lines passing between the free ends of the separating chromosomes, and quite distinct from the central fibres. These are the *Verbindungsfäden* of German authors, whose origin is undoubtedly to be found in the *linin*-thread of the nucleus.

The difference we observed, in the chromosomes in the case of prophasis, between the nuclei of the dividing individuals and those of the spore-forming ones, is also here discernible. Fig. 48 represents an individual in which the first two buds are just dividing; these are seen more magnified in Fig. 48 *A* and *B*. In both of them the nucleus is much elongated and dumbbell-shaped, with a large archoplasm at each end, represented as usual by rough granulations staining very deeply with aniline dyes. Scattered within the swollen ends of the nuclei are seen many chromosomes, each of which is distinctly discernible as a double row of minute microsomes; whereas in the nuclei of dividing individuals at about the same stage, the chromosomes consist of only a single row (compare this figure with Figs. 35 and 36). In Fig. 48, *A*. in which the ob-

ject is looked at a little from one side, the central fibres (*c. f.*) lying in the furrow of the nucleus, and the radiating fibres (*r. f.*) proceeding from the upper archoplasm, are distinctly visible. It thus seems that segmentation of the chromosomes in spore-formation takes place before the previous division of the nucleus is yet at its end. This is also to be looked upon certainly as the result of the rapid progress of the bud-formation.

4. *Anaphasis*:—The details of anaphasis can only be observed in individuals which divide, since in those which form spores the nucleus divides successively without the intervention of any resting stages, as mentioned above; and at the end of the spore-formation the diminished size of the nucleus makes it very difficult to observe the individual chromosomes with any accuracy. I will therefore only give my observations of the changes, at this stage, in the nuclein substance of the dividing individuals. Fig. 38 represents one of the nuclei soon after division. The parallel arrangement of the chromosomes, their relative position in regard to the archoplasm (*a*), and the general shape of the nucleus are all displayed very plainly. The positions of some of the chromosomes at its right lower corner, however, become more or less irregular in their relations. Fig. 40, which represents one of the nuclei nearly at the stage of Fig. 7, shows that the chromosomes have lost their parallel arrangement. The microsomes are, however, still very small, represented only by small dark-coloured dots. In Figs. 39-43 the microsomes have become somewhat swollen up, assuming nearly the characters of those in the resting nuclei. The microsomes in Fig. 42, however, appear to be a little too large, owing perhaps to the action of the acetic acid with which the specimen was treated.

5. *The fate of the archoplasm*:—The archoplasm, as we have stated, comes to be seen first at the stage a little before the division or the

spore-formation. During the whole process of the division of the nucleus it remains always closely attached to the outside of the latter, and in the dividing individual is still to be recognised just before the separation of the individuals, as represented by Fig. 12, and even for some time after the complete separation, as one of my preparations of the animal twelve hours after has shown very plainly. After the formation of the peristome, teeth, etc. it becomes indistinct. In the spore-buds the archoplasm remains till the final stage when the spore is ready to swim off from the body of the mother animal, as Figs. 23-30, 49-51 show very plainly. In nearly ripe spores, Figs. 27-**30**, the archoplasm lies on that side of the body which is attached to the mother cell, and is therefore at the head end of the spore when it detaches itself from the body of the mother. This is more clearly to be seen from sections of nearly ripe spores, represented by Figs. 23-26. I sought after this body in ripe spores treated with acetic-acid-methyl-green, but could not get any satisfactory knowledge of its existence. From the position of the archoplasm, and from the manner of division in the foregoing stages, it is clear that the free end of the spore corresponds to the equatorial part of the archoplasmic spindle, while the end attached to the surface of the mother animal is the pole of the spindle.

6. *The centrosome:*—In many cases in the centre of the archoplasm is seen a small round body, which often colours very deeply with aniline dyes, especially with such colouring matters as rubin, iodine-green-methylen-blue, etc., but also with haematoxylin, when this is used in conjunction with iron ammonium alum, as given by *Heidenhein* (**15**, p. 118-119). In a short note on the conjugation of Noctiluca (**26**, p. 3) I stated that at the poles of the two conjugating nuclei, deeply-staining round bodies are found, similar to those seen in the centre of the archoplasm, and suggested that such bodies

are the centrosomes so commonly observed at the poles of the spindle of the dividing nuclei of other animals. The body now in question is seen in Figs. 35, 39, 45, 49, 50, 51, (*c*). In Fig. 39 it is very plainly visible in the centre of the upper archoplasm as a small round body surrounded by a clear space. In the lower archoplasm of the same figure, it can not be seen so well, owing to its position beneath the nucleus, but is discernible by focussing the tube. In Fig. 44 two centrosomes are seen close together in the centre of the archoplasm. The nucleus to which this archoplasm belongs is, as stated above, in the condition just before division, while in Fig. 45 it is again seen as a single body. It is also seen in four cells in Fig. 49, as small dark-staining dots. This is also the case in Fig. 50, and in the archoplasm on the left hand side of Fig. 52. It is not visible in other figures, while in the archoplasm represented by Fig. 42 and more clearly by Fig. 48, there is seen a number of small bodies in place of the centrosomes. These bodies are not always quite spherical like the ordinary centrosomes, many of them being more or less elongated, and often presenting curved rods like those, described by *Hermann* (**20**, p. 585), in the spermatocyte of Proteus; they are perhaps to be looked upon as a group of centrosomes like those, described by *M. Heidenhein* (**16**, p. 54-68), in the lymphocytes of rabbits. In my previous paper I have given a case where the probable origin of the centrosome from the nucleus is shown. Since then I have met with no case similar to that one, but in another, I saw, at the side of a nucleus which had just divided, a small deeply stained body (*c*), close to the nuclear wall, (Fig. 38) probably representing the last stage of the disappearance of the centrosome within the nucleus.

B. GENERAL CONSIDERATIONS.

a. Division.

Cienkowsky, in his second paper on Noctiluca (**8**, p. 55), speaks of its reproduction by means of division and says :—" Sie wurde am vollständigsten von *Brightwell* verfolgt, dessen Beobachtungen ich so wohl an normal gebauten wie auch an eingekugelten Individuen bestätigen konnte." *Cienkowsky* thus believes with *Brightwell* (**5**, p. 10) that the division can take place not only in normal individuals but also in animals in which the tentacle, mouth, etc. have become lost. So far as I have observed. I can only confirm *Robin's* observations (**6**, p. 1064) that the division takes place only in individuals in which the tentacles have disappeared. The plane of division is longitudinal to the body of the animal, as *Robin* also states ; but the first spindle lies in the direction of the longitudinal plane, cutting the animal, therefore, transversely. The resulting nuclei separate from each other and finally come to lie opposite, as is also pointed out by *Robin.* In the dividing individual, the plane of division of the nucleus appears to lie in the longitudinal axis of the body from the first.

The primary dividing furrow makes its appearance along the dorsal median line of the continuation of the mouth and the *Staborgan*, and extends till the body has taken the form of a biscuit; but the complete separation of the body into two parts proceeds, according to my observations, from the periphery towards the central protoplasmic mass, which for a long time remains continuous, as may be seen in Figs. 10, 11, and 12. *Robin*, who studied very thoroughly the division of Noctiluca, has arrived, so I learn from *Bütschli* (**6**, p. 1063-1067), at conclusions which in the main are in accordance with mine; the only difference being in the manner of the final division of the

individuals, which, according to him, goes either from the periphery towards the central mass or *vice versa*.

b. Spore-formation.

The general phases of spore-formation have been worked over by previous writers, such as *Cienkowsky* (**7, 8**), *Robin* (**31**), and others. The spore-forming individuals appear to possess a less quantity of cell-substance than ordinary individuals, owing to the flowing together of the protoplasmic network with the central mass, as *Cienkowsky* (**7, 8**) rightly observed. The observations of the same naturalist of the partial division of the entire body in the earlier phases of budding, are, as stated above, in accordance with my own, although the lines of division are not so deep as *Cienkowsky* shows them in his Fig. 14 (**7**); but then division of the body into four parts is more seldom met with than that into two.

The two nuclei resulting from the division of the first nucleus do not usually remain stationary, but soon change their relative positions, and thus the second division does not generally take place at right angles to the first, as will be seen in Figs. 15, 16, and 48. In consequence of this, the first four nuclei very seldom lie regularly on the surface of the cell. The time of the division of the first four nuclei is also not exactly the same in all of them, considerable fluctuations being observed in this and also in all the succeeding stages. Accordingly, the number of the nuclei, as well as that of the spores, is not uniformly a multiple of two, as counted by *Robin* (**6**, p. 1069), but quite irregular. *Cienkowsky* (**7**, p. 135) and also *Robin* (**6**, p. 1070) admit, in fact, that the division becomes tolerably irregular when the number of the nuclei becomes large. The number of ripe spores also varies much according to the size of the individual, being less in smaller individuals than in larger ones. I have counted in one case more

than five-hundred spores in a very large animal, while in other cases the number was little over three-hundred. But this also is not always the case, for the area covered by the spores varies in different individuals, though in usual cases it is nearly one-half to one-third of the half surface of the cell.

We will now speak about some points of more or less interest in the phenomena displayed in the karyokinetic division of the cells of other animals compared with those presented in the above described cell-division of Noctiluca. These points concern the ARCHOPLASM, the CENTROSOME, and the SPINDLE FIBRES.

The ARCHOPLASM is, as will have been seen from the preceding description, first generally found quite near the nucleus when this is in preparation to divide, (see Figs. 1, 2, 13, 31 etc.). At this time the nuclear wall is always distinctly to be seen, and the size of the archoplasm as compared with that of the nucleus is such as to make its origin from the nucleoplasm very improbable. The nucleus at this stage and in the living condition, appears more or less homogeneous and transparent, while the archoplasm, consisting of coarse granules, is so conspicuous as to be easily mistaken for it. This fact perhaps explains the statements of *Cienkowsky* (**8**, p. 54), who supposes the nucleus to disappear at the beginning of the spore-formation in the animal. *Robin* (**6**, p. 1070) has observed the archoplasm and the changes it passes through in nuclear division, but seems to have mistaken it for the nucleus. In dividing individuals, the archoplasm remains closely attached to the nucleus and for a long time until the complete separation of the daughter individuals, as will be seen in Fig. 12; and in spore-forming animals up to the time when the spore is ready to swim away from the surface of the

body, the most remarkable thing here being the flagellum which takes its origin from the centre of the archoplasm, as already stated. *Strasburger* (**35**, p. 65) gives many similar cases in the swarm spores of many plants, where the cilia spring out of the *kinoplasm* (*archoplasm* of Boveri), or at least from the point where the last trace of the *kinoplasm* was found. We shall come back to this again soon.

Looking over the literature of the nuclear division of Protozoons, there is, so far as I know, as yet no case mentioned of the existence of the archoplasm and the centrosome, except in a short notice of mine (**26**, p. 3). The gathering of the cytoplasm and the conical bodies at the poles of the dividing nucleus was observed by *Schewiakoff* (**32**, p. 221) in Euglypha alveolata. But these appear at the opposite poles of the nucleus from the beginning, as he very explicitly says :—" Bevor noch die Abplattung des kugeligen Kerns stattfindet, sah man das Cytoplasma an zwei beliebigen entgegengesetzten Stellen, den zukünftigen Polen, sich anhäufen. Kurze Zeit darauf beginnt die Abplattung und man bemerkt gleichzeitig, dass die Kernwandung an diesen beiden Stellen in den Kern sich etwas einstülpt, wodurch zwei kleine Dellen gebildet werden. Auf dem Grund diese Dellen gewahrt man einen kleinen homogenen Höcker, dessen Umrisse, dank der starken Lichtbrechung, deutlich hervortreten. Besonders scharf treten sie bei abgetödteten Thieren hervor, und erscheinen als mattglänzende, gut begrenzte, ellipsoidale Körper, die von Farbmitteln nicht im mindesten tingiert werden." He justly compares this body with the polar corpuscles of *Ed. van Beneden*, and considers the hyaline part of the Spirochona nucleus, observed by *R. Hertwig*, to be in the same category. According to *R. Hertwig* (**23**, p. 156), the nucleus of Spirochona gemmipara consists of two parts, a larger granular part and a smaller hyaline part,

separated from each other by a transverse line. Stained with *Beale's* carmine, the former colours much more quickly than the latter. The same state of things was observed by this author in Leptodiscus medusoides (**24**, p. 311), the only representative of Cystoflagellata, other than Noctiluca, found by *Hertwig* at Messina. The nucleus consists here also of two parts, "einem feinkörnigen und einem homogenen." "In dem einen Falle war die homogene Kernsubstanz unverändert, dagegen fanden sich in der feinkörnigen grössere und kleinere Körperchen, die sich in Carmin dunkler färbten und offenbar eine bedeutendere Dichtigkeit besassen. In anderen Fall liess sich die Differenzierung in zwei Substanzen nicht nachweisen" (**24**, p. 312). The first change that occurs in the homogeneous part of the Spirochona nucleus before its division is the appearance of a small central corn (**23**, p. 161). This he calls a "nucleolus," which is very probably the free area around the centrosome, such as is represented in my Fig. 13. Some changes occur in the granular part also, and the nucleus soon assumes an elongated form with homogeneous masses placed at its poles. In this way the nucleus assumes a form in which five parts are distinctly to be made out: viz., two homogeneous terminal plates, two striated portions, and a granular median part, considered by *Hertwig* to correspond with the nuclear plate. There is thus a remarkable resemblance between the homogeneous part of the Spirochona nucleus and the archoplasm of Noctiluca, although the aspects of the two are different, the former looking transparent and the latter granular. But no one will doubt of this similarity, who compares my Fig. 14 or 52 with *Hertwig's* Fig. 17, *a*. What transformations the homogeneous part of the Spirochona nucleus undergoes during the division is not quite clear from *Hertwig's* descriptions. From what we get from the Noctiluca nucleus, it seems very probable that

the division commences in this homogeneous part. The median granular part of the spindle of the Spirochona nucleus, as is given in *Hertwig's* Fig. 17, *b* and *c*, is not to be seen in the Noctiluca spindle. It is, however, very probable that here the uniform length of the chromosomes causes the appearance of a distinct median portion. One great difference to be observed between the two, is that in Spirochona and Leptodiscus the archoplasm is so closely united with the nucleus that it appears as if it formed a part of it. Whether we have to consider the two nuclear parts as a single nucleus, or whether the granular part alone is to be taken as a nucleus is not quite clear, since no membrane is to be seen around the nucleus, although *Hertwig* believes in the existence of one from analogy with nuclei of other Protozoons. In Leptodiscus, however, *Hertwig* speaks of the existence of a distinct membrane around the two portions of the nucleus taken together. In this connection *Hertwig* says (**24,** p. 311-312):—"Der Kern des Leptodiscus stimmt somit vollkommen mit dem der Spirochona gemmipara überein. Wie ich in einer früheren Arbeit gezeigt habe, besteht der Nucleus dieses Infusors ebenfalls aus einer feinkörnigen und einer homogenen Substanz, die sich beide gegen einander mit einer glatten Contour absetzen und gemeinsam von einer zarten Kernmembran umhüllt werden." Although division of the Leptodiscus nucleus was not observed by *Hertwig*, the perfect similarity of its structure with that of Spirochona, makes it very probable that here also the homogeneous portion plays an active part in the cell-division and corresponds with the archoplasm of the Noctiluca nucleus.

The occurrence of the kinetic centres—the archoplasms—apart from the nucleus in Noctiluca, deserves special attention. *Richard Hertwig* (**25,** p. 106) in his very interesting lecture on fertilization and conjugation delivered in Berlin, speaking of the constant occur-

rence of the centrosome in Metazoa, makes the following remarks on it as regards the Protozoa:—

"Mit diesen für die Metazoen geltenden Verallgemeinerungen sind die Beobachtungen an Protozoen zunächst nicht in Einklang zu bringen. Meines Wissens ist nur für Noctiluca die Anwesenheit eines vom Kern unabhängigen Centrosomas von *Ishikawa* angegeben worden, und auch hier handelt es sich nur um eine Vermuthung. Wo sonst Protozoenkerne genauer auf ihre Theilung hin geprüft worden sind, hat sich herausgestellt, dass die activen Substanzen, welche die Kerntheilung veranlassen, in Inneren des Kernes liegen, und als Bestandtheile desselben angesehen werden müssen. Ich habe das Gesagte für Actinosphærium nachweisen können; bei Euglypha, welche nach *Schewiakoff's* Untersuchungen in der Kerntheilung mit Actinosphærium sehr übereinstimmt, scheint ein gleiches Verhalten zu herrschen. Am beweiskräftigsten sind aber die Nebenkerne der Infusorien, deren Theilung am auffälligsten unter den Protozoen an die Sporenbildung der Metazoen erinnert."

In Actinosphærium eichhorni and in Spirochona it is true that the kinetic substance of the nuclear division—the archoplasm—lies within the nucleus. In the micronuclei of all the Infusorians, as far as they have been studied by many eminent investigators, there is as yet no case known where extranuclear kinetic centres are proved to exist. The state of things appears, however, to be a little different in Euglypha alveolata, where the *Polkörperchen* lie in small concavities of the nuclear membrane at two opposite poles of the nucleus. Of the origin of the *Polkörperchen*, *Schewiakoff* (**32**, p. 222) says:—

"Verfolgt man aufmerksam seine Entstehung, so kann man die Vermuthung aussprechen, dass es, wenn auch theilweise, aus dem sich differenzirenden Cytoplasma gebildet sind." It, therefore, appears

that the kinetic centre—the *Polkörperchen*—of Euglypha is derived from the cytoplasm and not from the nuclear substance, just as in the case of Noctiluca. But while in Noctiluca the archoplasm appears to remain in the cytoplasm, in Euglypha it becomes intermixed with the nucleus, when the division of the latter is completed, as we see from the following words of *Schewiakoff* (**32**, p. 238):—"Vor allen Dingen wird das ehemaligen Polkörperchen vollkommen in die Kernsubstanz eingezogen; man gewahrt nichts mehr von den linsenartigen Hervorstülpung; wie es noch in den kurz vorhergehenden Stadien der Fall war, sondern der Kern erscheint einheitlich und besitzt eine regelmässige glatte Oberfläche." It thus appears, that the *Polkörperchen* here are formed from the cytoplasm at the beginning of the nuclear division, and become drawn into the body of the nucleus at the end of it. The *Polkörperchen* is therefore not a permanent body, but has always to be formed anew from a part of the cytoplasm at the beginning of each nuclear division. As I have no ground to doubt the accuracy of the very beautiful observations of *Schewiakoff*, I have only to conclude that the fate of his *Polkörperchen* is not exactly the same with that of the archoplasm of Noctiluca, nor do I doubt the observations of *Hertwig* and others of the kinetic centres lying within the nucleus. From all this it appears that the above statement of *R. Hertwig* as regards the position of the kinetic substance in Protozoa and Metazoa, holds true; that in Protozoa this substance lies within the nucleus and that in Metazoa it lies outside it. But it appears also that there is considerable fluctuation within the Protozoa. Of special interest in this connection is the position of the archoplasm in Noctiluca and in Leptodiscus—the two relatives of Cystoflagellata: for while in Leptodiscus this substance lies within the nucleus, in Noctiluca it lies outside it. Cystoflagellata thus forms, in this res-

pect, an intermediate stage, so to speak, between the Metazoa and other Protozoa. If in Euglypha alveolata the *Polkörperchen* is formed from the nuclear substance and becomes connected with the cytoplasm during the nuclear division, to be again withdrawn into the nucleus, it forms in that case an interesting connecting link between Leptodiscus and Noctiluca! But we must confine ourselves to facts only.

Another interesting thing about the Noctiluca archoplasm is its extreme similarity with the *Nebenkern* of *von la Valette St. George* (**13, 14**), and *Platner* (**29**). Just like the archoplasm of Noctiluca, the *Nevenkern* of these authors is formed by the consolidation of a part of the cytoplasm at one side of the nucleus, and when this becomes visible the "Schleifenbündel des Kerns ist stets mit der Spitze gegen jene Verdichtung hingerichtet" (*la Valette* **13**, p. 7). The part played by the *Nevenkern* appears, however, not quite clear, but *von la Valette St. George* considers it very probable that the spindle fibres are formed from its elements in Blatta, since these transform again to form the *Nebenkern* (**13**, p. 10). When this fact is proved, then the parallelism between the Noctiluca archoplasm and the *Nebenkern* of *von la Valette St. George* will be proved beyond all doubt. It will not be, however, a rash conclusion to consider these two bodies as identical, especially when we consider the fate of the *Nebenkern* in the formation of the spermatozoa, as will be seen later on. The close similarity between the two will also be seen by comparing *von la Valette's* Fig. 6 on Plate III of "*Kölliker's* Festschrift, 1887" with my Figs. 1, 2, 6, or 7.

Of not less interest than the archoplasm is the origin of the spindle-fibres. Two different opinions are here also to be met with as in the case of the origin of the kinetic centre. *Strasburger* in his earlier works (**33, 34**) as well as in his newest (**35**, p. 59-62), has shown in

many vegetable cells the cytoplasmic origin of the spindle fibres. In Spirogyra polytæniata a part of the cytoplasm—the kinoplasm—differentiates itself in the form of parallel fibres on both sides of the flattened nucleus, even when the nuclear wall is very plainly to be seen. Meanwhile the nucleus becomes flatter than before, and the chromosomes arrange themselves in the equatorial plane. The parallel fibres come now to be seen distinctly within the nucleus on both sides of the equatorial row of chromosomes. The fibres are, according to *Strasburger*, the same fibres as were seen outside the nucleus and have pushed into the nucleus through the sieve-like pores formed in the nuclear wall, and attached themselves to the chromosomes. The nuclear membrane disappears after this, and a perfect spindle with the equatorial row of chromosomes is formed (**34**, p. 11). In the nuclear division of the protoplasmic *Wandbeleg* of the embryo-sac of Leucojum æstivum, the same author describes the formation of the spindle out of the cytoplasm surrounding the nucleus at the time when the nuclear wall still exists. In the pollen-mother-cells of Lilium bulbiferum (**36**, p. 182-183), *Strasburger* describes the formation of the spindle as like that in Spirogyra. Here twelve fibres are thrown out from both the *Centrospheres* towards the centre of the nucleus and push into its interior, and then, the nuclear wall becoming indistinct, a perfect spindle is formed by the union of the free ends of the fibres at the equator. In his latest paper (**36**) *Strasburger* still holds his views as to the cytoplasmic origin of the spindle-fibres for plant cells. On the other hand, *Pfitzner* (**28**, p. 655), *Rabl* (**30**, p. 269), *Zacharias* (**39**, p. 852 ; **40**, p. 334), *O. Hertwig* (**22**, p. 163), and others, believe in the formation of the spindle fibres out of the nuclear substance. *O. Hertwig* on this point says (**22**, p. 163):—" Bei einzelnen Mollusken (Pterotrachea, Phyllirhoë) haben Fol und ich beobachtet, dass die Polspindel im

Innern des Keimbläschens, welches hier übrigens von geringer Grösse ist, angelegt wird, so lange noch die Kernmembran vorhanden ist. Die Annahme, dass in diesem Fall Protoplasma von aussen in den Kernraum hineingedrungen sei, will mir wenigstens als eine gezwungene erscheinen." While thus the views of *Strasburger* and *Hertwig* are quite different as to the origin of the spindle fibres, the former deriving them from the cytoplasm and the latter from the nucleoplasm, the opinions of *Flemming* (**12**, p. 75), *Platner* (**29**, p. 70), and others stand midway between, since according to these authors the equatorial part of the spindle is formed from the nucleoplasm, and the polar parts from the cytoplasm. The late investigation by *August Brauer* (**4**) of the spermatogenesis of Ascaris megalocephala is very interesting on this point, inasmuch as the origin of the spindle fibres is different in two varieties of the same species. In Ascaris megalocephala univalens the centrosome is found in the nucleus, and there forms a small spindle which this author compares with the central spindle of *Hermann*, although the origin of that is different, as will be shown later on. In Ascaris megalocephala bivalens, on the other hand, the centrosome is first found in the cytoplasm, closely attached to the nuclear wall. By the division of this is also formed a small spindle lying first tangentially to the surface of the nucleus, but the two centrosomes gradually separate from each other until they come to lie diametrically opposite to each other. What becomes of the central spindle fibres is not known, but the author is inclined to think that these are divided in the separation of the centrosome (**4**, p. 181).

Of no less interest than the above are the observations of *Hermann*, the result of whose investigation on the spermatocytes of Salamandra maculata, specially directed to the study of the origin of the spindle fibres, remarkably accords with that of mine upon the

Noctiluca spindle. In the cytoplasm of the spermatocyte at some distance from the nucleus, there is formed a small spindle, from the poles of which, when it has attained a certain size, a number of fibres is seen passing to the chromosomes. By growth and change of position, it gradually attains the ordinary form of the spindle with equatorial chromosomes. In this full grown spindle there are thus two sets of fibres : the one continuous between the two centrosomes and lying axially ; and the other passing from the centrosomes to the chromosomes. The latter set of fibres therefore lies more or less like a mantle above the other set, which he calls the *central spindle* (**20**, p. 580). When division of the cell is completed the fibres of the central spindle return to the protoplasm (?), while the other fibres collect together to form the archoplasms of the daughter cells. This description of *Hermann's* stands thus in beautiful correspondence with that of the formation of the spindle in Noctiluca, given above; but with the difference that in Noctiluca, from the persistence of the nuclear wall, the central spindle does not lie, strictly speaking, in the axis of the whole system, but in the groove formed on one side of the nucleus by the depression of its wall, and therefore the mantle-like fibres as well as the chromosomes do not lie quite around the axial spindle. This otherwise exact correspondence of the spindle fibres in these two widely different animals is, beyond all doubt, a very interesting phenomenon, and renders desirable, in my opinion, further investigations into the structure and formation of different kinds of cells in other animals and perhaps also in plants.

In this connection I must not leave unnoticed the question of the *Verbindungsfäden*, of whose origin there are also as many interpretations as there are questions on other matters. Many investigators, such as *O. Hertwig* (**21, 22**), *Boveri* (**3**), *Ed. van Beneden* and *Neyt* (**1**), and

Brauer (**4**), consider the *Verbindungsfäden* to be formed from the *linin*-substance by the separation of the chromosomes towards the poles of the spindle. According to *Strasburger* (**36**, p. 183), these are formed from the same substance as the kinetic fibres, which enter from the poles of the spindle and meet in the equatorial plane. This view accords well with that of *Hermann*, in so far that the latter author considers the central spindle as forming the *Verbindungsfäden* of other investigators. In spite of the great similarity between the structures of the spindle in Noctiluca and Salamandra this view of *Hermann's* concerning the *Verbindungsfäden* can not be applied in explanation of the same structure in the Noctiluca spindle, since, as we see from the above description, the persistence of the nuclear wall in Noctiluca naturally shuts off all possibility of confounding the fibres of the central spindle with those extending between the free ends of the separating chromosomes. Moreover, the two sets of fibres optically appear quite different from each other, as will be seen in Fig. 37, (*c.f.*) and (*r.f.*).

Of the origin of the radial fibres in Noctiluca, I can say but a few words, since the whole problem still remains very obscure and requires a thorough study with better methods and optical instruments.[1] In sections given above, these fibres, which are found within the nucleus and probably attached to the chromosomes, appear to come into close juxtaposition, but not to be continuous with those without, *i.e.*, those seen within and without the nucleus appear to be different from each other, the former originating from the nucleoplasm and the latter from the cytoplasm, just as *Brauer* thinks concerning the formation of the spindle fibres of Ascaris megalocephala bivalens (**4**, p. 182-183).

1. It may be here remarked that our climate is not fitted for the use of Zeiss apochromatic systems, the dampness of the air in the warm season soon bringing considerable alterations in the lenses.

Last of all, the part played by the archoplasm in the formation of the cilia in Noctiluca deserves some attention. *Strasburger* in his *Histologische Beiträge*, Heft IV., gives many similar cases of the formation of the cilia from the archoplasm (**35**, p. 62-132; **36**, p. 184). The observations of *Henking* (**17, 18**), *Platner* (**29**), and above all of *von la Valette St. George* (**13, 14**), of the part played by the *Nebenkern* in the formation of the spermatozoons are very interesting in this connection, since the *Nebenkern* of these authors (especially of *von la Valette St. George*) corresponds, as said above, in all particulars with my 'archoplasm'; and it is, therefore, very reasonable *a priori* to consider that that part of the cytoplasm which is especially differentiated for kinetic functions, is transformed to form a part of the tail of a spermatozoon or the flagellum of a Noctiluca spore.

C. *Summary*

1) The division of the animal is preceded by the loss of the peristome, teeth, and the tentacle, the last of which is not thrown off, as *Robin* is inclined to think, but is withdrawn into the body of the animal. The mouth and the *Staborgan* are, however, always present (*Robin*).

2) The spore-forming individuals differ from the dividing ones in not possessing the mouth and the *Staborgan* in addition to the organs above mentioned, and by the excessively empty appearance of the cell interior (*Cienkowsky*).

3) The division of the nucleus is always preceded by the concentration of part of the cytoplasm in the form of a spherical or oval granular body, mostly close to the nucleus. This is the archoplasm or the kinetic centre of division, and corresponds most probably with the *Nebenkern* of *von la Valette St. George*.

4) In living animals at the stage of (3), the nucleus appears more

or less homogeneous and transparent, and is not so distinctly to be seen as the archoplasm. But treated with reagents, the chromosomes come into view distinctly.

5) Each chromosome consists of a row of disc-shaped microsomes irregularly scattered in the nucleoplasm. The number of the chromosomes is not clear, but in most cases has been counted to be ten.

6) The chromatin substance of each of the microsome discs collects at the periphery and forms a microsome-ring.

7) In the nucleus of a dividing animal, each microsome-ring splits into half-rings thus dividing a chromosome in halves, while in that of the spore-forming animal two successive divisions of a microsome-ring take place, so that a single chromosome is directly divided into four daughter ones.

8) The chromosomes collect on that side of the nucleus which is nearest to the archoplasm, and spread out towards the other pole. The pole where the archoplasm lies thus corresponds to *Rabl's Polfeld* and the other pole to his *Gengenpol.*

9) The archoplasm divides and forms a very large spindle which first lies tangential to the surface of the nucleus. This division of the archoplasm is succeeded by the separation of the chromosomes into two groups each attracted (?) by its respective archoplasm.

10) The archoplasmic spindle thus formed pushes-in the nuclear wall on which it lies, and the nucleus assumes in consequence a half-ring form.

11) By the separation of the archoplasms, a spindle is produced which in all essential characters appears like the form known as the "disaster stage," with a large archoplasmic mass at each end of the spindle.

12) The fibres of this spindle are therefore continuous from one pole to the other and lying outside the nuclear wall become in no way

connected with the chromosomes. But there is seen at this stage another set of fibres running from the centre of the archoplasm to the polar ends of the chromosomes. This structure of the spindle corresponds exactly with that of the spermatocyte of Salamandra maculata, as investigated by *Hermann*, with only the difference of the persistence of the nuclear wall in Noctiluca, and the necessary modification in consequence of this fact. The optical appearance of these two kinds of fibres is different, just as in Salamandra.

13) Besides these two sets of fibres, the *Verbindungsfäden* are clearly to be recognised extending between the separating chromosomes.

14) The central spindle fibres originate from the archoplasm, the radial fibres probably from both the cyto- and nucleo-plasms, and the *Verbindungsfäden* from the *linin*-substance.

15) In the spore-buds the archoplasm is to be seen lying close to the nucleus up to the time of the full development of the spore just before its detachment from the mother animal, and a part of it becomes transformed into the flagellum, just as in many vegetable swarm-spores, as *Strasburger* shows.

16) In the centre of the archoplasm is generally seen a centrosome, which often shows a dumb-bell form. Sometimes, however, two centrosomes are found in the archoplasm of the spore-forming cells. In many cases, again, there is found in the centre of the archoplasm a number of small oval, rod-shaped or curved bodies, staining exactly like centrosomes, instead of one or two centrosomes. These may represent the group of centrosomes of *Heidenhein*.

17) The origin and the fate of the centrosome are not known. In a few instances it appears to be formed from the nucleus.

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List of References.

1. *van Beneden, Ed. et Adolphe Neyt.*—Nouvelle recherches sur la fecondation et la division mitotique chez l'Ascaride mégalocéphale. Bulletins de l'Academie royale de Belgique, 3e serie, tome XIV, no° 8; 1887.
2. *Bergh, R. S.*—Recherches sur les noyaux de l'Urostyla grandis et de l'Urostyla intermedia, n. sp. Archives de Biologie, tome IX; 1889.
3. *Boveri, Th.*—Zellen-Studien, Heft 2. Fischer, Jena; 1888.
4. *Brauer, Aug.*—Zur Kenntniss der Spermatogenese von Ascaris megalocephala. Arch. f. mik. Anat. Bd. XLII; 1893.
5. *Brightwell, T.*—On the self-division in Noctiluca. Quart. Journ. Micr. Sci. Vol V; 1857.
6. *Bütschli, O.*—Cystoflagellata in Bronn's Klassen und Ordnungen. I Bd. Abth. 2; 1889.
7. *Cienkowsky, L.*—Ueber Schwärmerbildung bei Noctiluca miliaris. Arch. f. mik. Anat. Bd. VII; 1871.
8. *Cienkowsky, L.*—Ueber Noctiluca miliaris Sur. Arch. f. mikr. Anat. Bd. IX; 1873.
9. *Flemming, W.*—Neue Beiträge zur Kenntniss der Zelle, I. Theil. Arch. f. mikr. Anat. Bd. XXIX; 1887.
10. *Flemming, W.*—Neue Beiträge zur Kenntniss der Zelle, II. Theil. Arch. f. mikr. Anat. Bd. XXXVII; 1891.
11. *Flemming, W.*—Ueber Zelltheilung. Verhandl. der anat. Gesellschaft zu München; 1891.
12. *Flemming, W.*—Zelle. Ergebnisse der Anatomie und Entwicklungsgeschichte. 1892.
13. *St. George, von la Valette.*—Spermatologische Beiträge, II. Mittheilung. Arch. f. mikr. Anat Bd. XXVII; 1886.
14. *St. George, von la Valette.*—Zelltheilung und Samenbildung bei Forficula auricularia. Festschrift für von Kölliker. 1887.
15. *Heidenhein, M.*—Kern und Protoplasma. Festschrift für von Kolliker. 1892.

16. *Heidenhein, M.*—Ueber die Centralkörpergruppe in den Lymphocyten der Saugethiere. Verhandl. der anat. Gesellschaft zu Göttingen; 1893.

17. *Henking, H.*—Untersuchungen über die ersten Entwicklungsvorgänge in den Eiern der Insecten. I. Das Ei von Pieris brassicæ L., nebst Bemerkungen über Samen und Samenbildungen. Zeit. f. wiss. Zool. Bd. XLIX; 1890.

18. *Henking, H.*—The Same. II. Ueber Spermatogenese und deren Beziehung zur Eientwicklung bei Pyrrhocoris apterus L. Zeit. f. wiss. Zool. Bd. LI; 1891.

19. *Henking, H.*—The Same. III. Spezielles und Allgemeines. Zeit. f. wiss. Zool. Bd. LIV; 1893.

20. *Hermann, F.*—Beiträge zur Lehre von der Entstehung der karyokinetischen Spindel. Arch. f. mikr. Anat. Bd. XXXVII; 1891.

21. *Hertwig, O.*—Vergleich der Ei- und Samenbildung bei Nematoden. Arch. f. mikr. Anat. Bd. XXXVI; 1890.

22. *Hertwig, O.*—Die Zelle und die Gewebe. Fischer, Jena; 1892.

23. *Hertwig, R.*—Ueber den Bau und Entwicklung der Spirochona gemmipara. Jenaisch Zeitschr. f. Naturw. Bd. XI; 1877.

24. *Hertwig, R.*—Ueber Leptodiscus medusoides. Jenaische Zeitschr. f. Naturw. Bd. XI; 1877.

25. *Hertwig, R.*—Ueber Befruchtung und Conjugation. Verhdl. d. deutsch. zool. Gesellschaft zu Berlin; 1892.

26. *Ishikawa, C.*—Vorläufige Mittheilungen über die Conjugationserscheinungen bei den Noctiluceen. Zool. Anzeiger; 1891.

27. *Ishikawa, C.*—Studies of Reproductive Elements. I. Spermatogenesis, Ovogenesis, and Fertilization in Diaptomus sp. This journal Vol. V; 1891.

28. *Pfitzner, W.*—Beiträge zur Lehre vom Bau des Zellkerns und seiner Theilungserscheinungen. Arch. f. mikr. Anat. Bd. XXII; 1883.

29. *Platner, G.*—Beiträge zur Kenntniss der Zelle und ihrer Theilungserscheinungen. Arch. f. mikr. Anat. Bd. XXXIII; 1889.

30. *Rabl, C.*—Ueber Zelltheilung. Morph. Jahrb. Bd. X; 1885.

31. *Robin, Ch.*—Recherches s. la reproduction gemmipare et fissipare des Noctiluques. Journ. anat. et physiol. 1878.

32. *Schewiakoff, W.*—Ueber die karyokinetische Kerntheilung der Euglypha alveolata. Morphologisches Jahrbuch. Bd. XIII; 1888.

33. *Strasburger, Ed.*—Befruchtungsvorgang bei den Phanerogamen Fischer, Jena; 1884.

34. *Strasburger, Ed.*—Histologische Beiträge, I. Kern und Zelltheilung im Pflanzenreiche. Fischer, Jena; 1888.

35. *Strasburger, Ed.*—Histologische Beiträge, Heft IV. Schwärmsporen, Gameten, pflanzliche Spermatozoiden, und, das Wesen der Befruchtung. Fischer, Jena; 1892.

36. *Strasburger, Ed.*—Zu dem jetzigen Stande der Kern- und Zelltheilungsfragen. Anat. Anzeiger VIII. Jahrg; 1893.

37. *Weismann, A. und C. Ishikawa.*—Ueber die Bildung der Richtungskörper bei tierischen Eiern. I. Ber. d. Naturf. Ges. zu Freiburg i. B. Bd. III; 1887.

38. *Waldeyer, W.*—Ueber Karyokinese und ihre Beziehungen zu den Befruchtungsvorgängen. Arch. f. mikr. Anat. Bd. XXXII; 1888.

39. *Zacharias.*—Ueber die Spermatozoiden. Ann. Bot. Ztg; 1881.

40. *Zacharias.*—Beiträge zur Kenntnisse des Zellkerns und die Sexualzellen. Bot. Ztg; 1887.

Explanation of Plates, XI-XIV.

a — archoplasm.
a sp—archoplasmic spindle.
cf — central fibres.
n — nucleus.
rf — radial fibres.
t — tentacle.
x — median part of the archoplasmic spindle left after the division of the nucleus.

$\frac{2}{B} \times Z$ = Drawn by means of camera lucida under Zeiss' objective B with eye-piece 2 : magnified 60 diameters.

$\frac{I}{III} \times S$ = Drawn by means of camera lucida under Seibert's objective III with eye-piece 1 : magnified 120 diameters.

$\frac{2}{1/12} \times Z$ = Drawn by means of camera lucida under Zeiss' homegeneous immersion $1/12$ with eye-piece 1 : magnified 350 diameters.

$\frac{4}{1/12} \times Z$ = Drawn by means of camera lucida under Zeiss' homegeneous immersion $1/12$ with eye-piece 4 : magnified 650 diameters.

Plate XI.

(*Division* of the *animal.*)

Fig. 1. Individual preparing to divide ; tentacle (*t*) much reduced in size ; archoplasm (*a*) very plainly visible as a granular spherical body lying close to the nucleus (*n*). Killed with Flemming's stronger fluid, and stained with Böhmer's hæmatoxylin.

Fig. 2. Another individual of about the same stage as that of fig. 1 ; in centre of archoplasm a centrosome surrounded by clear space. Similarly treated as above.

Fig. 3. Central plasm of a dividing individual with archoplasmic spindle (*a sp*) lying close to the nucleus ; chromosomes gathered towards pole of nucleus

where spindle touches it. Acetic acid methyl-green preparation.

Fig. 4. More advanced stage of division; nucleus now dumbbell-shaped with archoplasms at its poles; narrow line of protoplasm marking line of division of animal seen in front of and behind central protoplasmic mass; the latter also constricted into two portions. Acetic acid methyl-green preparation.

Figs. 5-9. Different stages of division, showing various positions of archoplasm in regard to those of nucleus. All acetic acid methyl-green preparations.

Plate XII.

(Figs. 10, 11, & 12 represent *division*; Figs. 13-19, *spore-formation*.)

Figs. 10 and 11. Further stages of division; tentacles already at stage of fig. 11. Killed with Flemming's stronger fluid, and stained with Böhmer's hæmatoxylin.

Fig. 12. Portion of dividing individual with narrow protoplasmic connection (*p c*) still visible; tentacles grown much longer. Drawn from a living animal.

Fig. 13. First stage of spore-formation; archoplasm (*a*) as a granulated mass lying close to nucleus (*n*); dumbbell-shaped clear space within archoplasm, which represents clear zone around a dividing centrosome. Drawn from living animal.

Fig. 14. Further stage of spore-formation about the same as that of fig. 4. Acetic acid methyl-green preparation.

Fig. 15. Still further stage of spore-formation; division of nucleus now nearly at an end, parts still connected by narrow bridge. Acetic acid methyl-green preparation.

Fig. 16. Central plasm and dividing nucleus at stage similar to that of last figure; drawn from living animal.

Fig. 17. Four-nucleated stage of spore-formation; archoplasms belonging to nuclei in state of division, each forming large spindle. Drawn from acetic acid methyl-green preparation.

Fig. 18. Side view of stage slightly later than that of fig. 17. One of the four nuclei, on right-hand of figure already divided into two daughter nuclei. (The two dumbbell-shaped nuclei, not represented in the figure, are also in a similar

stage of division to that represented on the left of the figure). Acetic acid methyl-green preparation.

Fig. 19. Stage similar to that of fig. 18, showing line of partial division of animal. Picro-acetic acid preparation without staining.

Plate XIII.

(Figs. 20-30, represent *spore-formation;* Figs. 31-37, *division.*)

Fig. 20. Stage of sporulation more advanced than that of fig. 19, showing also line of partial division of animal. (It will be observed that the number of the nuclei in each half is not equal, four lying on one side and two on the other). Acetic acid methyl-green preparation.

Fig. 21. Stage of spore-formation still later than above, showing also line of partial division, six nuclei lying on one side and four on the other. Acetic acid methyl-green preparation.

Fig. 22. Stage with fifty-one nuclei seen laterally; spore-cells on the other side represented by fainter shading. Acetic acid methyl-green preparation.

Figs. 23-26. Sections of spore-buds in last stage of division. Killed with Flemming's stronger fluid and stained with combination of Böhmer's hæmatoxylin and rubin.

Figs. 27-30. Ripe spores just before detachment from the mother-body. Killed with Flemming's stronger solution and stained with carbolic acid fuchsin.

Fig. 31. Part of an individual just preparing to divide; very large archoplasm close to nucleus; tentacle and teeth still visible. Killed with Flemming's stronger solution and stained with Böhmer's hæmatoxylin.

Fig. 32. Nucleus of individual preparing to divide. Killed with Boveri's picro-acetic acid, and stained with acid-fuchsin and methylen-blue.

Figs. 33 & 34. Nuclei of two dividing individuals, in which the longitudinal division of chromosomes are beautifully to be seen. Killed with Flemming's stronger fluid, and stained with acid-fuchsin and methylen-blue.

Figs. 35 & 35. A. Nucleus in process of division as seen at two different levels: fig. 35 at level of archoplasmic spindle, and fig. 35. A. at a lower level; centrosome within the archoplasm on the lower side. Killed with Flemming's stronger fluid and stained with Böhmer's hæmatoxylin.

Fig. 36. Stage like the last drawn at the level of archoplasmic spindle. CENTRAL- and RADIAL-FIBRES beautifully seen; owing to accumulation of dirty matters over archoplasms, centrosomes not to be recognised. Killed with Flemming's stronger fluid and stained with Böhmer's hæmatoxylin.

Fig. 37. Section of nucleus in division at about same stage as that represented by fig. 35 or 36; RADIAL-FIBRES in the homogeneous mass of nucleoplasm at lower pole of nucleus, different apearance from CENTRAL-FIBRES. Killed with Flemming's stronger fluid and stained with Böhmer's hæmatoxylin.

Plate XIV.

(Figs. 38-43, *division;* Figs. 44-52, *spore-formation.*)

Fig. 38. One of the nuclei of a dividing individual just after division; chromosomes still radiating from pole where archoplasm lies; small centrosome lying close to nucleus. Killed with Flemming's stronger fluid and stained with Böhmer's hæmatoxylin.

Fig. 39. Stage same as fig. 38. Centrosome surrounded by clear space at centre of upper archoplasm; in lower archoplasm hidden below nucleus and therefore not well seen; part of CENTRAL-SPINDLE in shape of a diagonal figure (*x*) between the separating nuclei. Killed with Boveri's picro-acetic acid solution and stained with Böhmer's hæmatoxylin.

Fig. 40. Another nucleus in anaphasis; microsomes plainly visible. Killed with Flemming's stronger fluid and stained with Böhmer's hæmatoxylin.

Fig. 41. Nucleus at stage slightly later than that of fig. 40. Microsomes no longer in rows, but more or less irregular and thus assuming the position in the resting nuclei. Killed with Flemming's stronger fluid, and stained with acid-fuchsin and methylen-blue.

Fig. 42. Similar nucleus treated with Boveri's picro-acetic acid and stained with acid-fuchsin and methylen-blue; microsomes very much swollen by action of acetic acid.

Fig. 43. Nuclei of a dividing individual at about the stage fig. 7 or 8. Killed with Flemming's stronger fluid and stained with acid-fuchsin and methylen-blue.

Fig. 44. Archoplasm and nucleus of spore-forming individual at about same stage as that represented by fig. 13; the two centrosomes plainly seen at side of nucleus within archoplasm. Killed with Flemming's stronger solution and stained with acid-fuchsin and methylen-blue.

Fig. 45. Similar stage to above. Single centrosome surrounded by a clear space and many fine fibres seen in the archoplasm next to nucleus; chromosomes gathered up at pole of nucleus facing archoplasm. Killed with Flemming's stronger fluid, and stained with acid-fuchsin and methylen-blue.

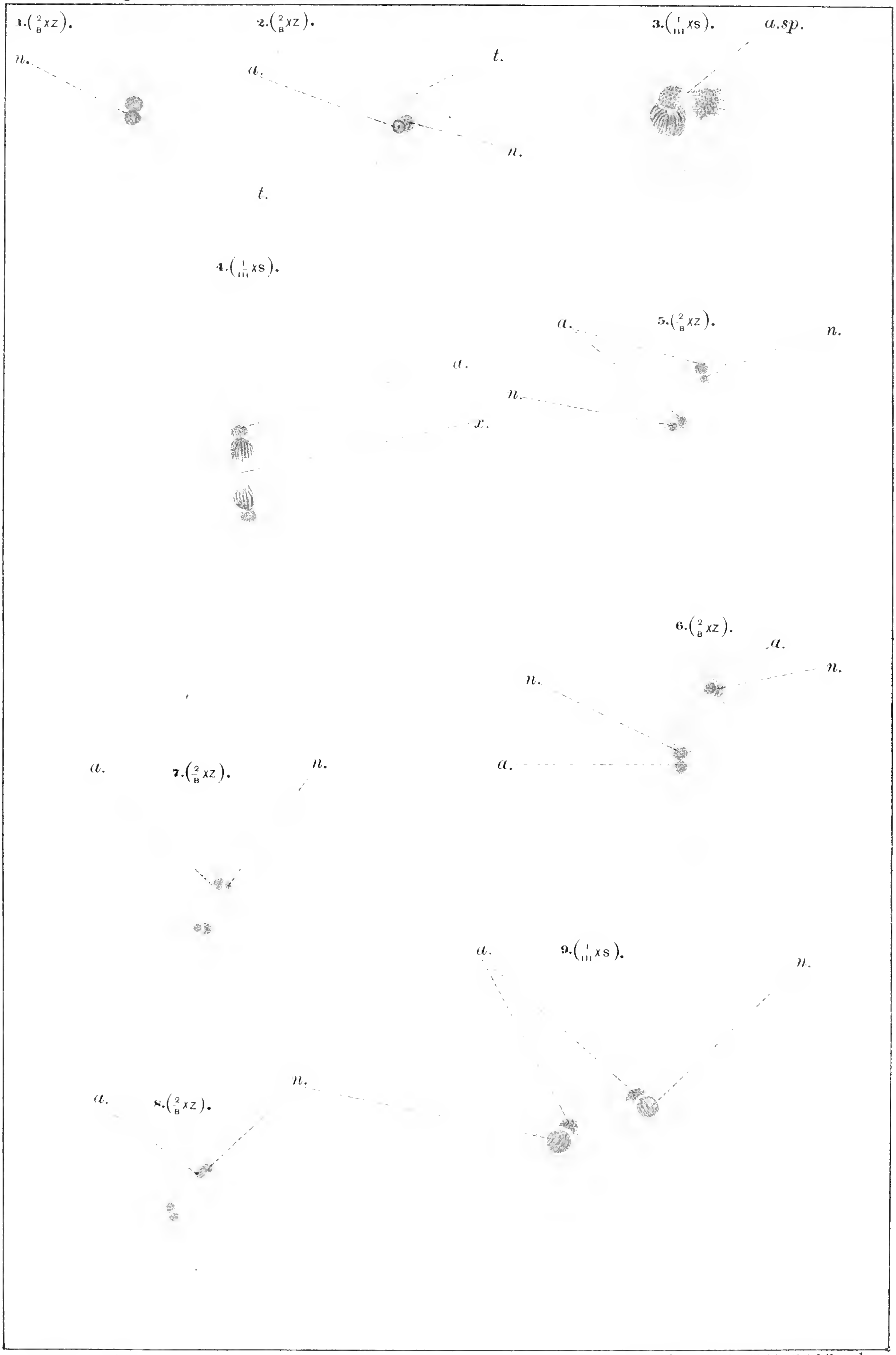
Fig. 46. Similar stage to fig. 45. Here the archoplasm lies at a lower level than nucleus. Treated as above.

Fig. 47. Nucleus of another individual at about same stage as above. Many of the chromosomes showing very plainly longitudinal divisions into four rows of microsomes. Treated as above.

Fig. 48. Two spore-buds in division; represented more highly magnified in fig. 48 A and B. Chromosomes within poles of nuclei longitudinally divided already at this stage; many darkly-staining bodies within archoplasms, probably corresponding to group of centrosomes observed by Heidenhein. Killed with Flemming's stronger fluid, and stained with acid-fuchsin and methylen-blue.

Fig. 49-51. Divisions of spore-buds; centrosomes in many of them. Killed with Flemming's stronger fluid, and stained with acid-fuchsin and methylen-blue.

Fig. 52. Abnormal division of spore-bud into three daughter cells. Treated as above.



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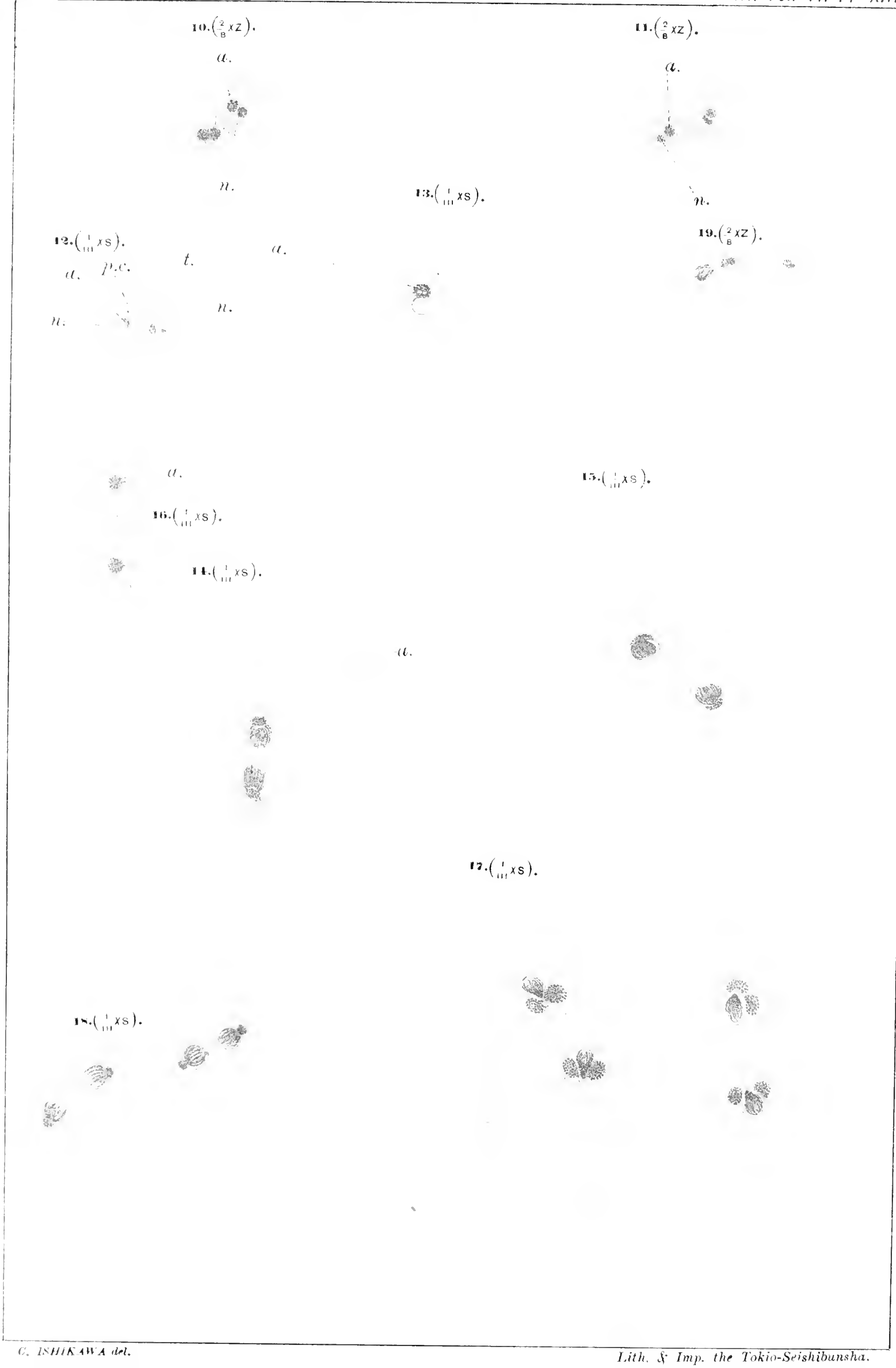

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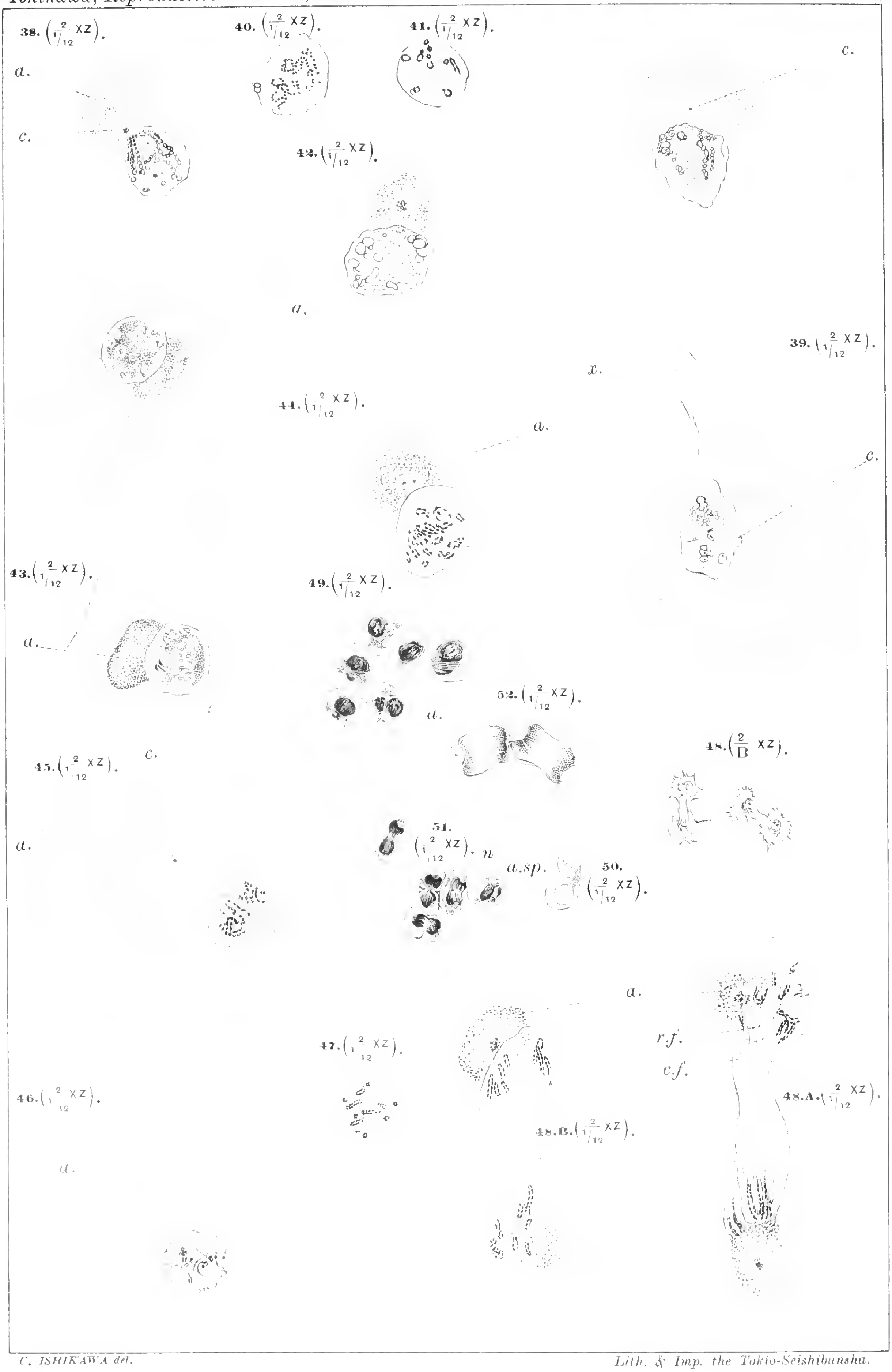

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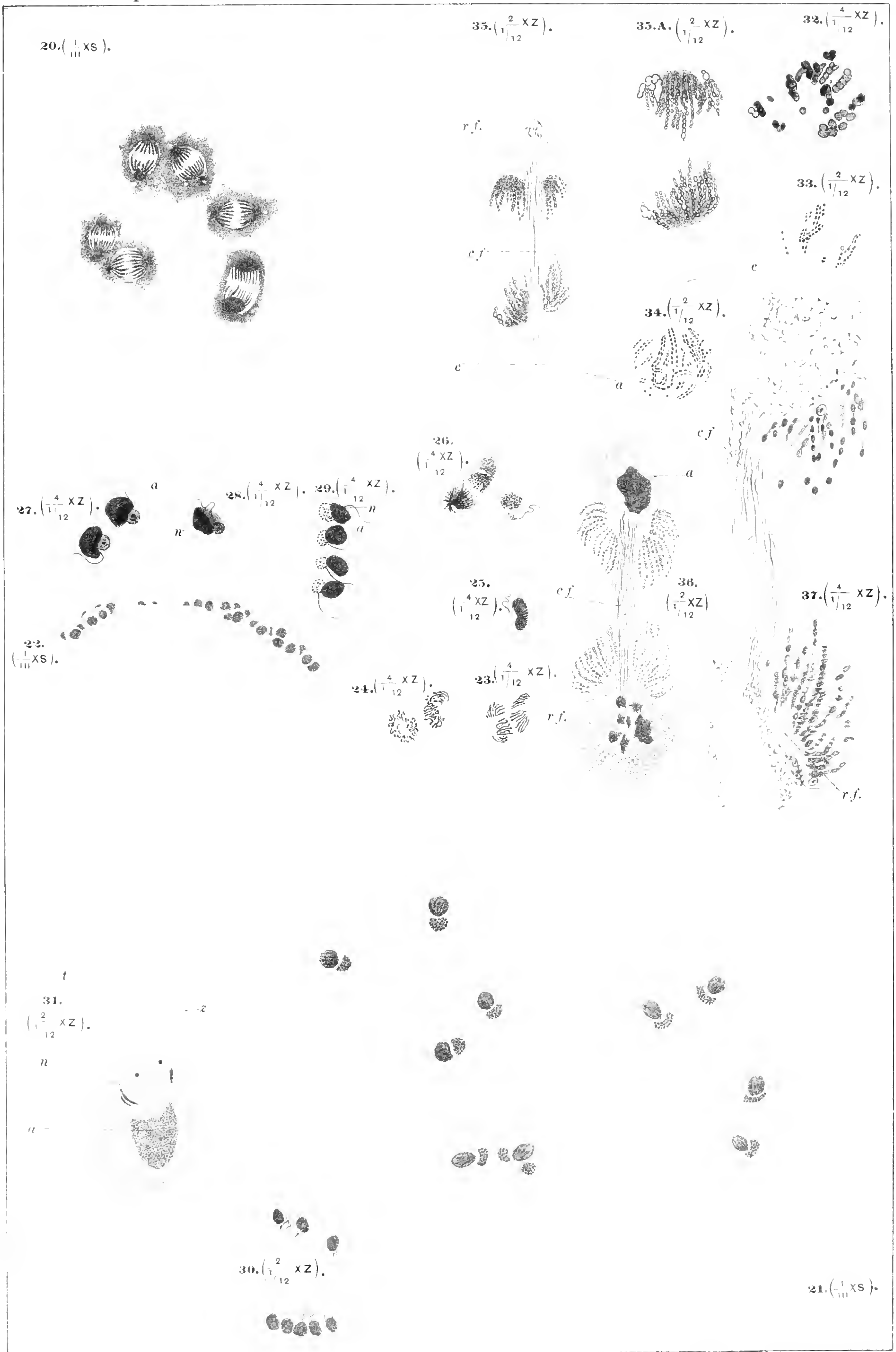

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On the Sero-Amniotic Connection and the Foetal Membranes in the Chick.

By

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Zoological Institute, Imperial University, Japan.

With Plates XV-XVII.

CONTENTS.

There is probably no other animal whose development has so much been made the subject of study not only by students in general but by special investigators as that of the chick, and, yet, it is a remarkable fact that erroneous notions prevail at the present day in

regard to some fundamental points in the structure and relations of the foetal membranes of the chick. It is generally assumed by writers on embryology that several amniotic folds, rising over the embryo, come together in the median line above the embryo, the edges of which entirely coalesce and whose cavities become continuous from one side to the other. The classical series of diagrams given in Foster and Balfour's Elements of Embryology (2nd ed., pp. 28–33), and copied very extensively, is constructed on this supposition. Some authors, as Schenk,[1] Kölliker, and Hertwig, have, it is true, remarked that the cavities of the lateral folds are separated by a longitudinal partition (named 'Amnionnaht' by Kölliker) for some time after the amniotic cavity is closed; but they also state that shortly afterwards, the partition breaks down, so that the cavities of the former lateral folds become directly continuous and the amnion proper entirely isolated from the serous envelope. In reality the facts are very different. While the horse-shoe shaped anterior amniotic fold grows backwards over the dorsal surface of the embryo, the cavities of the lateral folds are not in coalescence in the median line, but are from the first separated from each other by a partition which never breaks down. This is quite similar to the state of things in Chelonia recently described by Mitsukuri,[2] who gave the name of "Sero-Amniotic Connection" to this partition. This sero-amniotic connection undergoes in the chick some histological changes in the course of the development of the embryo, but remains as a partition to the last day of the incubation. This persistence of the sero-amniotic connection exercises, as in Chelonia, a great and complex influence on the foetal membranes in later stages. At the suggestion of Profs. Mitsukuri and Ijima, I have

1. S. L. Schenk: Beiträge zur Lehre von Amnion. Arch. f. Mikros. Anat. Bd. 7. 1871.
2. K. Mitsukuri: On the Foetal Membranes of Chelonia. Journ. of the College of Science, Imp. Univ., Japan. Vol. IV., Pt. 1. 1890.

been engaged in elucidating these points in the history of the foetal membranes of the chick and have brought to light some facts which appear to me very remarkable. The present article contains the main results of my investigation.

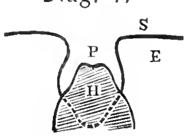
Before going further, I wish to express my best thanks to my teachers, Profs. Dr. Mitsukuri and Ijima, for their constant encouragement and advice throughout this investigation.

1. *Origin of the Sero-Amniotic Connection.*

If we examine a chick embryo of 25–30 hours (Diag. 1) we find that the extra-embryonic coelomic cavities (E) have already extended themselves on both sides, some distance in front of the embryo itself. These anterior parts of the extra-embryonic coelomic cavities have not, however, yet extended themselves directly in front of the head, and leave here a sort of bay (P), where the hypoblast and the epiblast alone are as yet present.

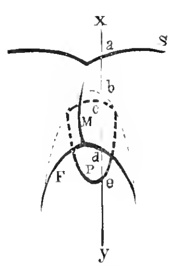
When the embryo has become 30–40 hours old (Diag. 2), the anterior limbs of the extra-embryonic coelomic cavities meet in front of the head, cutting off the previous bay from the region in front. Thus an area (P), which is formed by the hypoblast and the epiblast only, has been circumscribed just in front of, and beneath, the head. This is

Diag. 1.



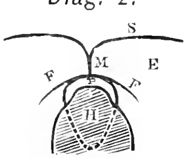
25-30 Hours.

Diag. 3.



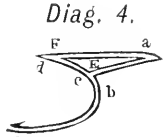
40-45 Hours.
Head not represented

Diag. 2.



30-40 Hours.

Diag. 4.



Longitudinal section through a line, x y (Diag. 3). Head not represented.

S.—Sinus terminalis.
P.—Proamnion.
H.—Head.
E.—Extra-embryonic coelomic cavity.
M.—Mesoblastic septum.
F.—Amniotic fold.

the Proamnion.[1] The extra-embryonic coelomic cavities which have thus met in front of the Proamnion, do not fuse together for a time, but have a median mesoblastic septum (M), consisting of two sheets of cells, each limiting the extra-embryonic coelomic cavity of its own side. At this stage the head fold of the amnion is raised along a line which passes through the anterior limit of the proamnion (F. F.).

When the embryo is 40–45 hours old (Diag. 3), the head is somewhat sunken down in the yolk sac, and the head fold of the amnion has grown posteriorly over the anterior part of the head. Thus the head is received into a cavity which ends blindly in front and is open posteriorly (Diag. 4). The dotted line of Diag. 3 shows the limits of this cavity. The mesoblastic septum is now obliterated near the *sinus terminalis* but the backward growth of the amniotic fold has prolonged it in the posterior direction (M). The proamnion (P), into which the extra-embryonic coelomic cavities gradually penetrate from both sides, is now out of sight, concealed beneath the head and the amniotic fold. In Fig. 1. and Figs. 57–59, the surface view and the sections of the mesoblastic septum are represented.

In Chelonia the extra-embryonic coelomic cavities secondarily insinuate themselves from both sides into the amniotic fold, which is at first purely epiblastic (See Figs. 2–4 and Figs. 17–19 of Mitsukuri's article). In the chick the extra-embryonic coelomic cavities, which are of course bounded by the mesoblast, spread themselves, from the very beginning of folding, almost coextensively with the epiblast, or in other words, nearly up to the posterior horse-shoe shaped edge, so that from the initial period, the dorsal part of the amniotic fold of the chick does not contain an extensive area free from the mesoblast,

1. E. Ravn: Ueber die Mesodermfreie Stelle in der Keimscheibe des Hühnerembryos. Arch. f. Anat. u. Physiol. (Anat. Abth.). 1886.

such as has been observed in Chelonia. But, still, we see the similarity between them in the process of folding. From the time when the amniotic fold has come to enclose some of the anterior part of the head, the epiblast seems to grow backwards always just a little in advance of the mesoblast, forming at the posterior edge a mass of epiblastic cells in the median line. Thus in Fig. 1 and Fig. 60 we see that a mass of the epiblastic cells is inserted between the two sheets of the mesoblastic septum. The same state is also observed in Chelonia when the amniotic fold has advanced considerably over the dorsal part of the embryo. Consequently the chick differs from Chelonia only by the fact that the early or solely epiblastic stage of the folding is abbreviated,—and with respect to this point it is highly interesting to notice that in the chick the amniotic fold is raised much later than in Chelonia.

The wedge-shaped epiblastic cell-mass near the posterior edge, inserted between the mesoblastic sheets, directly connects the epiblastic layer of the amnion and that of the serous envelope, and thus is seen at this stage quite the incipient condition of the sero-amniotic connection. Finally, the sero-amniotic connection lies in the direct prolongation of the line of the mesoblastic septum formed in front of it.

2. *Prolongation of the Sero-Amniotic Connection.*

From the time when the first trace of the sero-amniotic connection has appeared, the wedge-shaped mass of the epiblastic cells in the posterior edge of the head fold (referred to in the previous section) is enormously thickened in the dorso-ventral direction especially near the median line. From this place the mass is gradually attenuated postero-laterally along the horse-shoe shaped edge (Figs. 1–3, 38–40 and 62–65). Consequently along the posterior edge of the amniotic fold there is formed a delta-shaped epiblastic mass, which marks the

posterior end of the sero-amniotic connection. The extra-embryonic coelomic cavities of the two sides are separated by this delta. In a transverse section through the delta we see that the sero-amniotic connection is a very thick bridge between the outer and the inner epiblastic layers of the fold, and that the bridge is laterally constricted from both sides by the mesoblastic sheets, so that the connection is bi-concave (Fig. 60).

As the posterior edge of the amniotic fold with its epiblastic delta grows constantly backwards, and the extra-embryonic coelomic cavities as constantly encroach on the epiblastic mass of the posterior edge, a sheet of cells (Fig. 90, *s. a. c.*) connecting the amnion and the serous envelope is always left between the mesoblastic sheets of the two sides, and thus the sero-amniotic connection or the 'Amnionnaht' is prolonged backwards, so long as the posterior edge grows backwards. This goes on until the head fold meets the tail fold of the amnion. The delta also grows somewhat larger in absolute size in the course of the backward progress (Figs. 38–42, 60, 62, and 66). While these processes are being carried on, the curvature of the embryo takes place (Figs. 1–5) and the sero-amniotic connection also becomes curved at the same time.

The mesoblastic septum, which has been formed in front of the sero-amniotic connection, is gradually obliterated, and the extra-embryonic coelomic cavities of the two sides become continuous in front of the epiblastic sero-amniotic connection (Figs. 2–4).

Finally, in Chelonia the epiblastic delta is observed in the posterior edge in comparatively later stages of the folding, but in it the delta is not thickened dorso-ventrally so much as in the chick.

3. *Limit of the Prolongation of the Sero-Amniotic Connection.*

At the end of the second day or early in the third day of the incubation, a new amniotic fold arises behind the tail end as the primitive tail fold (Figs. 2 and 61). The process of its folding is quite similar to that of the first stages of the head fold. But its horse-shoe shaped edge is directed anteriorly and there is never found any trace of the mesoblastic septum which has been observed in the head fold. Early in the fourth day, while its cavity is still very shallow, it becomes fused with the head fold at the level of the still rudimentary right hind limb (Figs. 3 and 4).

The seeming growth of the tail fold in the anterior direction is due, in my opinion, not so much to actual advancement in that direction as to the backward growth of the tail and its consequent sinking into the yolk sac. If the head fold of the amnion arose a little earlier or grew backwards a little more rapidly than it actually does, it would reach the posterior end before any tail fold had risen and the case would be somewhat similar to that of Chelonia.

When the tail fold approaches the head fold, its anterior epiblastic edge becomes thickened into an irregular cell-mass, especially developed in the median line. Thus the elliptic opening which marks the point of the final closure of the amniotic cavity, or the 'Kommunikationsöffnung' of Schenk, is surrounded by a thick epiblastic ring, anteriorly deltoid and posteriorly irregular (Figs. 3, 4, 41, and 67). By degrees, as the epiblastic ring becomes thickened, the diameter of the opening is reduced until the folds have entirely coalesced, usually in 75–85 hours (Figs. 4, 5, and 42). Fig. 89 is a transverse section through the opening, when it has just closed, and when the epiblastic ring is greatly swollen.

We thus see that the prolongation of the sero-amniotic connection is stopped by the closure of the amniotic cavity. This is the period of the maximum development of the *epiblastic sero-amniotic connection.* At this time it has (Fig. 42) in a dorsal view the appearance of a slender streak slightly curved, with a concavity on the right side, and a swollen epiblastic mass at the spot where the amniotic folds have coalesced (Figs. 5 and 42). From this time on, although cells in it may multiply by division, it does not seem to enlarge as a whole—at any rate, in any great degree—but rather grows less and less, being encroached upon by mesoblastic cells, in a manner to be explained hereafter, until by the tenth day of the incubation it has been entirely replaced by a *secondary mesoblastic connection.*

4. *Changes in the Extra-Embryonic Germinal Layers.*

Every one who has studied the development of the chick well knows, that from very early stages the epiblast of the embryo proper is split up into cell-strata. The uppermost layer is built of flattened cells, while the under layer or layers are composed of slightly columnar cells. The stratification is not confined to the embryo proper, but gradually extends centrifugally into the epiblast of the amnion. Fig. 90 is taken from the dorsal part (for its exact locality see the explanation of figures) of a normal embryo of 78 hours, whose amniotic cavity has just been closed ; and in this figure we see that the stratification has advanced considerably beyond the lateral angle (A) of the amnion, but that there are as yet no distinct layers near the sero-amniotic connection.

The extra-embryonic parts of the coelomic cavity are at first lined by a single epithelial layer of mesoblastic cells. From an early period, however, a reticular tissue gradually spreads itself outwards

from the body wall on that side of the amniotic mesoblast which is turned towards the epiblast in the manner already described by Schenk.[1] This net-work advances hand in hand with the stratification of the epiblast, and the mesoblastic net-work also scarcely reaches the sero-amniotic connection before the amnion closes up (Fig. 90).

The stratification of the epiblast and the reticulation of the mesoblast spread farther in later hours of the incubation. At the end of the fourth day, the mesoblastic net-work has already reached the sero-amniotic connection and passed along both sides of it towards the serous envelope over which it also gradually spreads (Figs. 71 and 73). Distinct stratification of the epiblast is also found from the same period in the amnion as well as in the serous envelope (see the black layers in any of Figs. 70–87). Thus, in Fig. 71, which is of this stage, the previous *epithelial cells* of the coelom on both sides of the connection are no longer distinct, and their place is taken by an ill-defined reticular tissue which is intimately applied over the compact epiblastic bridge.

These new changes in the connection are very important, for they are steps to the formation of the secondary sero-amniotic connection, which will be explained in the following sections.

It is necessary to say here a few words about the allantois. This organ, which first appeared between the amnion and the serous envelope, on the right side of the embryo, about the time when the amnion had closed itself, has in this stage come near the sero-amniotic connection. As far as it has expanded itself, its vascular mesoblastic tissue has fused, on its way, with the mesoblastic epithelium of the serous envelope; and so the mesoblastic net-work, which has spread over the serous envelope, some distance beyond the connection is on

1. Loc. cit.

the right side directly traceable to that of the allantois (Figs. 5, 6, and 93).

5. *Replacement of Cells in the Sero-Amniotic Connection.*

I have mentioned in Section 2, that as the head-fold grows backwards, the epiblastic delta of its posterior edge is constantly, though only partially, being removed by the insinuation of the extra-embryonic coelomic cavities of the two sides; and that the remnant of the delta is prolonged backwards as the 'Amnionnaht' or the sero-amniotic connection. This process of insinuation prevails from the very beginning of the formation of the delta, and is continued after the amniotic cavity is closed. The mesoblastic net-work, having reached both sides of the connection, begins to pierce across the remnant epiblastic cell-bridge, so that at length the mesoblastic cells of one side meet with those of the opposite side, as we see in Figs. 72 and 74.

From the sixth day the epiblastic bridge is broken up here and there, beginning with the anterior part of the connection (Figs. 71, 72, 73, 74, and Figs. 6–7). In the posterior part of the connection, where the amniotic cavity has finally closed, remains of the epiblastic cell-mass, which there was the last to form, are found for a long time as irregular patches, as in sections (Figs. 76–78 and Fig. 8). Sometimes, the epiblastic cell-mass appears in sections as isolated patches imbedded in the mesoblastic tissue, as in Fig. 78, but by tracing them in other sections they are found to become continuous with the epiblast either of the amnion or of the serous envelope. By the ninth to the tenth day of the incubation, the epiblastic bridge has been completely replaced, along the entire length of the connection, by mesoblastic cells (Figs. 48–50).

The epiblast of the amnion and of the serous envelope are now entirely discontinuous. But the amnion and the serous envelope,

in both of which the epiblast and the mesoblast are by this time inseparably attached, are still connected along the line corresponding to the length of the sero-amniotic connection by means of insinuated mesoblastic net-work. Thus the *mesoblastic* or *secondary sero-amniotic connection* is formed.

6. Expansion of the Mesoblastic Connection.

The simple replacement of the epiblastic bridge by the mesoblastic net-work is not the sole event in the formation of the secondary sero-amniotic connection. From the nature of the mesoblastic connection its breadth is of course invariably greater than that of the previous epiblastic connection. But, over and beyond this, the mesoblast of the amnion and that of the serous envelope begins after the first week of the incubation to coalesce for a certain distance on both sides of the connection, so that in ten to twelve days, the connection is very much widened and assumes the form of a plate, viewed from above. Figs. 43–50 represent the breadth of the connection at different stages, each projected on a straight line. Figs. 76, 81, and 84 are transverse sections of the secondary connection, through the approximately middle level of the connection represented in Figs. 45, 49, and 50 respectively.

If we examine sections of embryos in their seventh to tenth day of the incubation, the process of coalescence is unfailingly witnessed. The plate-like mesoblastic connection seems to be produced by the intimate contact of mesoblastic cells (Fig. 75) in the narrow space on both sides of the preexisting connection, similar to the coalescence of the allantois with the serous envelope (Section 4).

The breadth of the secondary connection may, however, vary to a certain extent in different individuals, and not necessarily be proportional to the age of the embryo (Figs. 43–50).

In some individuals a peculiar triangular sheet is observed at the anterior end of the connection (Fig. 11). This is due to the following circumstance. The anterior part of the connection often obtains a considerable dorso-ventral extension, and thus produces a triangular sheet whose base corresponds to its anterior free edge, and whose sides correspond to the lines along which it joins respectively the amnion and the serous envelope (Figs. 11 and 79, Figs. 6 and 70). This triangular sheet does not stand edgewise, but comes to lie on its surface and hence is seen from outside, as in Fig. 11.

Thus, the mesoblastic bridge, which is supposed by all writers on the embryology of the chick to be absorbed, making the extra-embryonic coelomic cavity continuous from one side to the other, is in reality greatly widened and remains to the last day of the incubation.

7. *Perforation in the Mesoblastic Connection.*

From the eleventh day, the plate-like connection begins to be perforated here and there by numerous round and elliptic pores, which put the amniotic cavity in communication with the space outside the serous envelope, so that the plate is now changed into a sieve, and the epiblast of the amnion and that of the serous envelope again become continuous. This state of the connection is represented in Figs. 51–53. Each of these figures is of a view from inside the amnion, part of which is represented in circular outline (Amn.); a tube-like structure attached to this piece of the amnion is a part of the albumen sac which is lined by the serous envelope. How the albumen sac has assumed such a shape will be explained fully farther on. It is represented as filled with a blue mass. The membrane which separates the amniotic cavity from that of the albumen sac is the sero-amniotic connection. It has many perforations which con-

nect the two cavities, and appears sieve-like. Figs. 85–87 are from a series of sections through a perforated sero-amniotic connection of the twelfth day. In Fig. 85, which is the most anterior represented, the connection is entire and intact. In Fig. 86 it is very thin. In Fig. 87, there is a perforation.

These pores in the plate-like connection grow larger in size in later days, while the number becomes less by the running together of some adjacent ones. At length, after the sixteenth or seventeenth day the two sides of the plate are connected only by a few strands which remind us of the *chordae tendineae* of a mammalian heart (Figs. 54–56).

Thus the epiblast of the amnion and that of the serous envelope again become continuous; the extra-embryonic coelomic cavities of the two sides are as completely separated from each other throughout the entire extent of this sero-amniotic connection as in any previous stage.

8. *Supplement to the Preceding Sections.*

I will now make some necessary remarks on other points which it was convenient to withhold until I had traced, in the preceding sections, the main course of the changes in the connection itself.

In the extra-embryonic part, near the sero-amniotic connection at least, the epiblastic cells can usually be distinguished from the mesoblastic cells by the following points. In form, the mesoblastic cells are flattened in the plane of the layers themselves, wherever the cells are densely disposed, and are stellate, wherever the number of cells is very small, while the epiblastic cells are mostly perpendicular to the plane of the layers (Figs. 75, 80, 85, and 90). Consequently the mesoblastic tissue is schistose or reticulated, while the epiblastic layers are in more or less regular strata and compact.

On this account the two kinds of germinal layers are differently stained in the extra-embryonic region.

These differences are, however, not apparent in all cases. In later stages, both the layers are intimately united and in some places they can not be satisfactorily distinguished from each other. One may even come to doubt whether the epiblastic bridge is replaced by the mesoblastic tissue. But I have many preparations in which there can not be any doubt on this point. For instance, in the sections given in Figs. 71 and 72, taken from the stage represented in Fig. 6. in which the epiblastic bridge has begun to be invaded by the mesoblast, the one (Fig. 71) has the epiblast bridge still intact, while in the other (Fig. 72), which comes behind the first, there is distinctly no trace of the bridge, although it must have existed here earlier. The same state of things is observed in the case represented in Fig. 7 and Figs. 73–74. From these cases we may be certain that the epiblastic cells have been entirely withdrawn from the previous bridge and that the netted tissue is purely mesoblastic, as I have mentioned in Section 5.

That, near the posterior edge of the head fold of the amnion, an epiblastic cell-mass is produced was observed by Schenk in 1871. Of the fate of this mass he says: "So bin ich geneigt anzunehmen, dass die Zellen, welche die Verdickung ausmachen, bis zum einem bestimmten bleibenden Theile derselben nach und nach in Liquor Amnii aufgeben, und ihre Zerfallsprodukte scheinen zum guten Theile an der Embryonalleibe ihre Verwendung zu finden." This interpretation induced me to look carefully for the supposed processes of dissolution, but I have been unable to confirm his idea for the following reasons.

I have examined sections of nearly three dozen specimens of embryos from the second to the tenth day of the incubation, and

have been unable to find a single case like that represented in Fig. 1 of Schenk. On the contrary, the epiblastic cell-mass seems to me to have always a definite boundary on the amniotic side. Often we find a coagulum well stained with borax carmine, attached to the cell-mass, but a similar coagulum is found in other spaces, such as the central nervous canal, the alimentary canal, and also in the coelomic cavity. In Fig. 88 we see the cell-mass projecting into the amniotic cavity, and in Fig. 76 the remnant of this projecting mass is represented. These are, however, extreme instances and in ordinary cases the cell-mass, as a whole, is only slightly sunken into the amniotic cavity (Fig. 66). Such sinking into the amniotic cavity is not to be wondered at when we consider the weight of the mass itself.

As I have described in Sections 2 and 5, the epiblastic cell-mass is being constantly formed and constantly removed from the very beginning of the formation of the connection, and being situated near the free edge of the head fold, the product of its dissolution, if it were actually dissolved, would diffuse widely through the surrounding albuminous fluid, and but little of it be mingled with the amniotic fluid, which is speedily increased in volume, evidently from other sources.

Finally, from the physiological point of view, it seems to me to be improbable that the epiblastic cells dissolve up for the use of other epiblastic cells of the same surface.

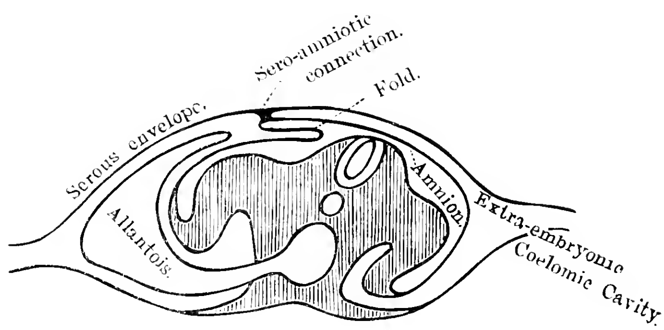
From the preceding discussion, and from the fact that from an early stage of folding the deltoid cell-mass is constantly added to at the posterior free edge and constantly reduced near its anterior apex, it is probable that the cell-mass is used up for the extension of the surface of the fold, and not for any other special purpose. It is, besides, a noteworthy fact, that in Chelonia the deltoid area is not thickened dorso-ventrally and undergoes, of course, no dissolution.

9. *Effects of the Sero-Amniotic Connection on the Foetal Membranes.*

About the time when the amniotic cavity is closed the allantois appears on the right side of the embryo in front of the level of the still rudimentary right hind limb; and it spreads itself gradually between the serous envelope and the amnion.

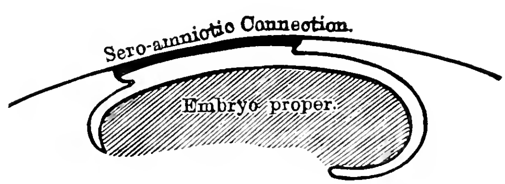
The first structure which we notice, and which seems to owe its origin to the presence of the sero-amniotic connection, is a curious infolding of the amnion near the connection, given in Diag. 5. It is

Diag. 5.



Transverse section of an embryo through the allantois and the fold

Diag. 6.



Longitudinal section through the sero-amniotic connection.

always on the right side of the connection and is pushed on towards the left. There takes place no folding before the allantois appears, and

the longitudinal extent of the fold depends on the length of the sero-amniotic connection. The form and extension of such a fold, vary greatly, however, in different individuals and at different stages. One instance of great development is given in Fig. 37, and its sections in Figs. 91–93. The planes of these sections are indicated by dotted lines in Fig. 37; in which the points, A, B, and C, on the dotted line marked 92, correspond to the points marked by the same letters in Fig. 92. It is not clear what significance this fold has.

At both extremities of the sero-amniotic connection the amnion is also slightly folded longitudinally (Diag. 6); transverse sections of such folds are represented in Fig. 68 (at the anterior end) and Fig. 69 (at the posterior end). In these figures, above the general amniotic cavity there appears a second smaller cavity marked *Amn.* A reference to Diag. 6 will show that this appearance is due to the folds. However, these transverse and longitudinal folds, being disturbed by the growth of the amnion as well as by the changes of the sero-amniotic connection itself, become insignificant after the first week of the incubation.

These temporary effects upon the amnion are, however, followed by others much more important on the allantois and also on the albumen-sac. The albumen-sac of certain birds, and the yolk sac of the chick have been studied by Duval[1] and recently by Virchow,[2] but knowledge of the general development of the foetal membranes of the chick is still left very imperfect. Hence, it may not be superfluous to give here a brief outline of the development of the foetal membranes

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1. M. Duval: Études histologiques et morphologiques sur les Annexes des Embryons d'Oiseau. Journ. de L'Anat. et de la Physiol., XX. 1884.

2. H. Virchow: Der Dottersack des Huhnes. Internationale Beiträge zur Wissenschaftlichen Medizin. Bd. 1. I regret that I know this article only by extracts given in Ergebnisse der Anat. u. Entwickl. (1892; Wiesbaden), and Zoologischer Jahresbericht zur 1891 (1893; Berlin).

themselves, in order that the influence of the sero-amniotic connection upon them may be better understood.

When at the end of the third day or early in the fourth day, the allantois appears on the right side of the embryo, between the serous envelope and the amnion, it is a discoidal sac with circular outline. At the end of the fifth day we can distinguish in a surface view the right and left allantoic arteries and one large allantoic vein in the allantois (Fig. 7). The right artery, which is always bifurcate in the inner limb of the allantois, supplies the posterior part of the allantois, while the vein is divided into two or three main branches in the outer limb. The left artery is smaller than the other two vessels, and is destined to supply the anterior part of the allantois.

From this stage the allantoic vein begins to indent the previously circular margin of the allantois at the point where the vein passes from its outer to its inner limb. The obstruction by the vein restrains and retards the growth of the allantois at this point, although it does not stop it entirely. Contiguous to this point, therefore, the allantois grows faster than at the point itself and thus an indentation is produced. With the growth of the allantois, the indentation becomes of great depth in later days, and the lobe of the allantois which lies in front of it comes gradually to overlap the lobe posterior to it, beginning from the apex of the indentation (Figs. 8, 9, and 10).

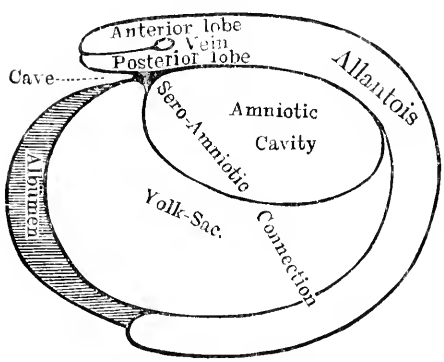
The right and left arteries also have a tendency to restrain the free growth of the allantois at the points where they pass from the inner to the outer limb. The effect is, however, quite trivial in comparison with that of the allantoic vein.

It is noteworthy that there is an intimate relation between the allantoic vein and the sero-amniotic connection. The allantoic vein always restrains the free growth of the allantois at a point a little to

the right of the connection (Figs. 8, 9, and 11). Whether the sero-amniotic connection, towards which the allantois spreads itself, is able to persist by the presence of the allantoic vein, or the margin of the allantois is specially indented by that vein on account of the presence of the sero-amniotic connection, is a question which can not be answered at present.

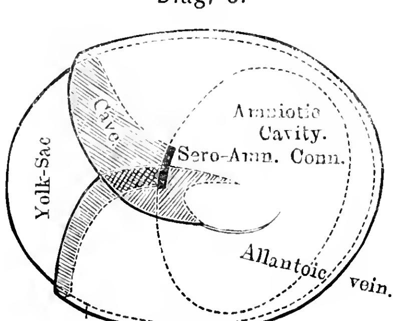
The overlapping of the allantoic lobes which began at the apex of the indentation and spread gradually peripherally (Fig. 10) progresses without any hindrance, until it reaches the sero-amniotic connection. As this offers an impassable obstacle, the allantoic lobes with their serous envelopes rise above the level of the sero-amniotic connection, and pass beyond it, projecting like the eaves of a roof. Diag. 7, which is a longitudinal section through the plane of the overlapping, shows how the allantois has risen above the sero-amniotic connection and passed beyond. There is thus produced, above the yolk sac and below the allantois, a sort of cave, at whose blind end is the sero-amniotic connection. Diag. 8 gives a surface view of these points. The part marked with lines slanting to the left shows the area of overlapping, while the part marked with lines slanting to the right

Diag. 7.



A Longitudinal Section.

Diag. 8.



A Surface View.

Egg in the 9th Day of Incubation.

shows the extent of the cave. The line at which the cave comes in contact with the amnion represents the sero-amniotic connection.

The two lobes of the allantois that overlap do so with their accompanying serous envelope, so that there ought to be sheets of epiblastic cells between the lobes, with the mesoblast and the hypoblast lining them. In point of fact, it becomes impossible to distinguish these layers, and only a simple septum, whose nature is not clear, divides the cavities of the two allantoic lobes. The epiblastic sheet is visible only near the outer limit of the septum (Fig. 82). In future I shall call this the '*Interallantoic Septum.*' It has a free edge along which the allantoic vein runs, and is, for the time, somewhat heart-shaped. At this stage other two minor interallantoic septa are formed by the right and left allantoic arteries in their respective lobes (Fig. 12). These are much smaller than the primary septum but in other respects show no essential difference. The amniotic cavity is now much enlarged and the yolk-sac much reduced. In fact, it is at this stage that the epiblastic sero-amniotic bridge has become almost entirely replaced by the mesoblastic tissue. Fig. 12 represents the surface view of the foetal membranes on the ninth day and Fig. 13 the vicinity of the sero-amniotic connection. In this particular example the amniotic cavity is drawn out towards the connection, and the cave (Fig. 13 DABC) is narrower and longer than that in Diag. 8.

With further growth, the amniotic cavity becomes enlarged, the yolk and the albumen-sac diminished, the allantois extended, and the interallantoic septum widened. Through these changes, occurring simultaneously, the foetal membranes reach in later days a degree of complexity, which well nigh defies description. I will only attempt here to explain the arrangement of the foetal membranes in an embryo of the tenth day by means of several figures and diagrams which are

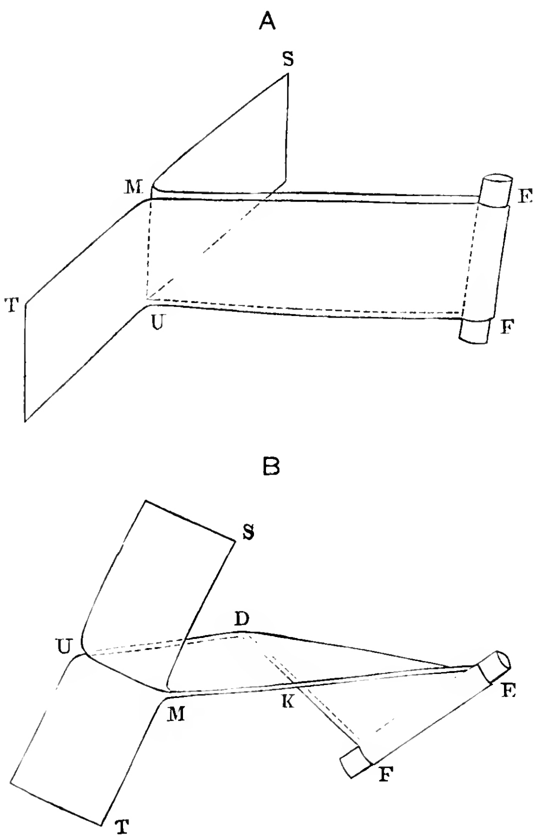
made so as to supplement one another. Fig. 14 is a surface view of the membranes when the shell is removed, in which, as usual, the allantois is coloured green. Fig. 15 gives the outlines of the membranes with the folds in the allantois shaded. These two figures are made exactly alike so that they may be superposed. Fig. 16 gives the vicinity of the sero-amniotic connection on an enlarged scale. Fig. 17 is intended to give the diagrammatic representation of the allantois with its folds and the interallantoic septum. In Fig. 18 the position of the amnion, the yolk-sac, the albumen-sac, and the sero-amniotic connection is given in addition. In the last two figures the outer limb of the allantois is represented as extensively removed, so as to show the inside of the allantoic cavity, and, in addition, the small end of the egg is figured as nearly cut off from the main portion of the egg and reflected to the left. In all the figures the egg is in one position, the anterior lobe of the allantois occupying the top of the figures and the posterior lobe, overlapped by it, appearing in the lower portion.

At this stage, the allantoic lobes have not yet closed entirely over the small pole of the egg so that the albumen-sac peeps through for a small space between them at that end. How the albumen-sac has thus come to lie within the allantoic lobes is easily understood if we imagine to ourselves the process explained in Diagrams 7 and 8 to continue until the allantoic lobes have extended themselves to the small pole of the egg. Within the space thus enclosed by the allantoic lobes, the three structures: the amnion, the yolk-sac, and the albumen-sac, are disposed, roughly speaking, in three stories (Fig. 18). The amnion with the enclosed embryo proper occupies the uppermost story at the larger pole of the egg. Below it comes the yolk-sac rather discoidal in shape, and then the albumen-sac. The perpendicular *x y.* (Fig. 18) axial to the three stories does not coincide with the axis of the egg through its poles, but is inclined to it as shown in the

figure. The albumen-sac, separated from the amnion for the most part by the intervening yolk-sac, sends out over the latter a hollow cave-like diverticulum towards the amnion (see Figs. 18 and 16), with which it comes in contact. The contact plate at the bottom of this diverticulum is the sero-amniotic connection (S. A. C., Fig. 18 and 16: sections in Figs. 80 and 81). In fact, it is the presence of the sero-amniotic connection that has produced the peculiar cave-like diverticulum. Let us now turn to the allantois itself. To understand the complexities arising in this membrane, it is necessary to bear clearly in one's mind the facts which have been brought out in the preceding pages; (1) that the allantoic vein in passing from the inner to the outer limb of the allantois causes an indentation in the margin of the allantois, retarding its growth at this point, and the lobe of the allantois in front of the indentation comes to overlap the lobe posterior to it, the area of overlapping being continuously increased with age; (2) that a terminal part of the left allantoic artery and a branch of the right allantoic artery produce respectively a minor indentation in the margin of the allantois. We must therefore know well how these bloodvessels run in the allantois. This may be learned by comparing carefully Fig. 14 and 15. First notice in Fig. 14 where all the allantoic vessels come out together from the umbilicus (for the exact position of which see also Fig. 17), the allantoic vein runs from the umbilicus to the point F (Fig. 15), along the inner limb; then from the point F gradually ascends along the line F E to the outer limb on which it emerges at the point E. The line F E must therefore be regarded as that along which interference with growth takes place, and the shaded areas F D E, and E C M D E (Fig. 15, for the present disregard the area A B C) together represent the area of overlapping of the two allantoic lobes, or in other words the extent of the interallantoic septum. The reason

why this is in two areas F D E and E C M D E is that the interallantoic septum has been folded in two along the axis E D. A reference to Diag. 9 will help to make this as well as some other points clear. Lettering in the diagram is as far as possible the same as in Figs. 15 and 17. In A, the rod F E, which represents the allantoic vein in its passage from the inner to the outer limb, is enfolded by a piece of paper so that it stands at the head of a deep fissure or incision (M E F U). U F E M S represents the anterior allantoic limb, U F E M T the posterior limb, and E M U F the interallantoic septum. In B, the area E M U F is folded along the line E D so that it appears now as two areas, FDE and E M U D E. The triangular area E D K is two layered, and the corresponding area in Figs. 17 and 18 is painted black. The line EM corresponds to the line E C M in Fig. 15 and E C H in Fig. 17. The line F D U corresponds to the line F D M in Fig. 15 and F D G in Fig. 17. I hope I have now made clear the disturbances caused by the allantoic vein and the relations of the interallantoic septum.

Diag. 9.



Let us now turn to the two arteries which cause disturbance in the allantois. One of them is the terminal part of the left allantoic artery. This artery, emerging from the umbilicus, runs along the inner limb towards the right in Fig. 14. It leaves the face represented in that figure on the right side low down, and running along on the other face, it enters the figure again at the right side above. It runs all this while along the inner limb and continues to do so until it reaches the point A in Fig. 15. From this point, it begins to rise along the line A B and emerges on the outer limb at the point B. The terminal portion of the left allantoic artery in thus passing from the inner to the outer limb causes the same sort of disturbance as was made by the allantoic vein, but to a smaller extent, and produces the fold A B C (in Figs. 15, 16, 17, and 18) which is marked on the outer allantoic limb by the line C B. This is therefore a secondary disturbance of the interallantoic septum.

Another minor septum is produced by a branch of the right allantoic artery. This artery proceeds from the umbilicus towards the left in Fig. 14 along the inner limb until it reaches the point Z (Fig. 15), giving off branches on its way. At the point Z it begins to leave the inner limb and rising gradually along line Z Y reaches the outer limb at the point Y, from which it is distributed on that limb. There is thus produced a mesentery-like fold, Z Y S,[1] of the allantois embracing the vessel along the line Z Y. (Cf. Diag. 9, A.) A secondary complication is produced by the fact that the surface of the fold is slightly curved, and is two layered in the area W Z Y (Fig. 15).

There are many individual variations in the configuration of the primary and minor septa, but the egg explained above in full may be taken as the typical one, and all the other cases can be ex-

1. A minor septum, caused by another branch of the right allantoic artery, which produces still further complication in this septum, is here, for the sake of simplicity not referred to.

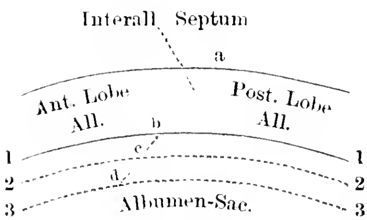
plained as modifications of this form. Thus, in Figs. 19, 20, and 21, from an embryo of the eleventh day which correspond respectively to Figs. 14, 15, and 16 of the preceding stage, the terminal portion of the left allantoic artery which runs along the edge A B in Fig. 15 and 17, runs along the axis on which the interallantoic septum is folded, and consequently the septum formed by the left artery alone is very small (ADK Fig. 21). Besides, in this example, the right allantoic artery, running along the inner allantoic limb, is divided at the point Z into two branches which, running along the free edges ZY and ZX, emerge on the outer limb at the points Y and X respectively, and thus a somewhat complicated minor septum ZSYXZ (see also fig. 20 *bis.*) is produced by that artery.

By the eleventh or twelfth day the allantoic lobes come together at the vegetable pole of the egg. Consequently, the albumen-sac or the '*sac placentoïde*' of Duval is entirely cut off from the surrounding space, as Duval says but contrary to Virchow's observation, while its cavity is lined by an epiblastic layer which has been continuous with that of the outer limb of the allantois. The length of all the allantoic septa is determined by the closure of the albumen sac, and all come together at that point where the allantoic lobes finally close.

It is at this stage that the perforation of the plate-like connection, before referred to, begins and it is a remarkable fact that for a few days after, the amniotic fluid soon coagulates in alcohol or in Kleinenberg's picro-sulphuric acid, just like the fluid in the albumen-sac. This seems to be due to the presence of albumen which has found its way through the perforations into the amniotic fluid.

As fast as the albumen is absorbed the primary interallantoic septum widens itself inwards at the expense of the inner limbs of the two allantoic lobes, which progressively coalesce with each other as the albumen diminishes and the albumen-sac contracts in volume. A reference to Diag. 10 will make what is meant clear. Suppose the outer limit of the albumen-sac to be the line 1-1. It is bounded by the two lobes of the allantois, and the area within which they overlap (a b) is the interallantoic septum. If now the albumen-sac decreased in diameter to the line 2-2, the interallantoic septum would then be widened to the point c, by the coalescence of the inner limbs of the two allantoic lobes. Thus the decrease of albumen will make the width of the interallantoic septum increase and the diameter of the albumen-sac grow smaller. Hence, the line along which the interallantoic septum is continuous with the inner allantoic limb (appearing as the points b, c, d, in a cross-section like Diag. 10), will always lie on the albumen-sac. The line AA represented in Figs. 51–56 is, in fact, this line. The relation of the inner edge of the interallantoic septum (DKG) in Figs. 17 and 18 to the cave-like diverticulum of the albumen-sac will make this evident. These changes are accompanied by gradual enlargement of the amniotic cavity and gradual decrease of the yolk. The stage in which these conditions are observed is given in Figs. 22–25, of which the first two represent surface views of the foetal membranes from two opposite sides, and the last two give their outlines. The letters as far as possible refer to the same parts as in Figs. 14–18. The bloodvessels first become visible in passing from beneath the amnion

Diag. 10.



Part of a Cross-Section of the Albumen-Sac.

(Figs. 23 and 25): the allantoic vein and the left allantoic artery on one side and the right allantoic artery on the other (Z, Fig. 25). The allantoic vein passes towards the right along the inner limb (Fig. 23) to the point marked F on Fig. 25, then passes from the inner to the outer limb along the line between F (Fig. 23) and E (Fig. 24)* (like the line FE, Fig. 15) and emerges on the outer limb at the point E (Figs. 22 & 24). The full line EO corresponds to the line EM in Fig. 15 and the fold FEOMF to the fold FECMDF in Fig. 15. The right allantoic artery starting from the point Z (Fig. 25) runs towards the left, but soon divides mainly into two branches (Fig. 23), each of which together with a third smaller one produces a fold in the allantois thus making three minor septa. No septum formed by the terminal portion of the left allantoic artery is present in this case. All the septa above referred to come together at the point marked O (Fig. 24). Fig. 26 represents the allantoic septa which have thus come together, as seen from the vegetable pole. Full lines show the tracings of these septa on the outer limb of the allantois. In addition to the primary interallantoic septum (E′O) there are three minor septa formed by three branches of the right allantoic artery which run along the free edges ZX, ZY, and MN. Notice also how the albumen-sac, now pitcher-shaped, has decreased in size.

By the sixteenth day yolk, as well as albumen, is much reduced in volume and the principal part of the egg is now the amnion occupied by the embryo proper (Figs. 27–30). Fig. 31 represents the septa (only three in all in this case) come together at the point P, as seen from the vegetable pole. In this figure we see that the right allantoic artery is

* The reader will understand that in tracing a bloodvessel or other structure from the surface represented in Figs. 23 and 25 to that given in Figs. 22 and 24, the upper edge of the former two figures should be applied to the lower edge of the latter two figures and also that as eggs have a considerable thickness, a structure leaving one surface may not be found at exactly the corresponding point of the other.

divided into two branches at the point Z (cf. Figs. 14 and 23), and that the two branches ascend directly, along the free edges ZY and ZX, towards the outer limb. At this stage the albumen-sac has assumed a peculiar elongated form, and the '*ombilic ombilical*' of Duval, by which the albumen-sac is united to the yolk-sac, has become conspicuous and easily observable. The sac is supplied with numerous blood vessels on the yolk side, and albumen is absorbed through that singular placental structure discovered by Duval in the linnet. The position of the umbilicus may vary in different individuals (Figs. 29 and 34), but the papillated placental structure of the albumen sac is always confined to the vicinity of the umbilicus. It is to be added that from this stage the blood vessels of the outer allantoic limb undergo certain changes in their distribution. One basal branch of the allantoic vein, which runs with the right allantoic artery, becomes conspicuous by anastomosing with other branches of the same vein (Figs. 27, 28, 32, and 33).

On the eighteenth day, the albumen-sac appears as a slender tube which opens into the amniotic cavity through the perforated connection (Figs. 32-35). Fig. 36 represents the septa come together at the point P as seen from the vegetable pole. In this case the spot where the right allantoic artery divides is concealed beneath the yolk-sac, and the two branches ascend along the free edges ZX and ZZ'Y respectively towards the outer allantoic limb. In this example the primary interallantoic septum has not been folded along an axis in the allantoic cavity (cf. Diag. 9, B), and moreover is not influenced by the left allantoic artery.

At length, the albumen becomes entirely absorbed and the vitelline membrane which had previously been imbedded in albumen is alone left in the former albumen-sac, often partly projected into the amniotic cavity through openings in the sero-amniotic connec-

tion. The yolk-sac becomes then enclosed by the abdominal wall and the entire space of the egg becomes filled by the amnion which is occupied by a fully grown embryo.

Contrary to the observations of some writers, the left allantoic artery is found to persist, though faintly, till the last day (Figs. 27, 28, 32, and 33), while the allantoic vein and the right allantoic artery are of great functional importance throughout the allantoic life. The inner allantoic limb is always supplied by much fewer blood vessels than the outer limb.

Finally, the sero-amniotic connection is quite without relation to the emergence of the embryo, which pushes out its beak into the air chamber through the point of the membrane far distant from the connection.

10. Comparison between the Sero-Amniotic Connection of the Chick and that of Chelonia.

The sero-amniotic connection of the chick, as described above, and that of Chelonia, as studied by Mitsukuri in *Clemmys* and *Trionyx*, may be compared as follows :—

The essential points in which they agree.—a). In the chick, there is found at the posterior edge of the amniotic fold a constant deltoid area free from the mesoblast. The same structure is observed in Chelonia in comparatively later stages of the progress of the fold backwards.

b). As long as the amniotic fold grows posteriorly the sero-amniotic connection or the remnant of the epiblastic deltoid area is equally prolonged posteriorly until it becomes a long string.

c) The sero-amniotic connection is more or less widened on both sides of the epiblastic bridge by the entrance of the netted meso-blastic tissue.

d). The epiblastic layers of the extra-embryonic parts are stratified, at least near the connection.

e). The connection between the serous envelope and the amnion persists during egg-life and no direct continuity is ever made between the extra-embryonic coelomic cavities of the two sides along the length of the connection.

f). The growth of the allantois is greatly influenced in later stages by the presence of the sero-amniotic connection.

The essential points in which they disagree.—a). In the chick the amniotic fold arises in comparatively later stages than in Chelonia, and the extra-embryonic coelomic cavity is from the first extended backwards in the amniotic fold almost as far as the epiblast has progressed, while in Chelonia the mesoblastic cavities insinuate themselves secondarily from the two sides in the latero-median direction. It is owing to this difference that in the amniotic fold of the chick there is found no conspicuous area free from the mesoblast, and that there is no mesoblastic septum in Chelonia before the epiblastic sero-amniotic connection appears.

b). In Chelonia the epiblastic delta of the amniotic fold is always larger than that of the chick, but in the latter it is enormously thickened dorso-ventrally.

c). In Chelonia the amniotic fold passes over the tail end before the latter is yet differentiated and there is therefore formed no proper tail fold. In the chick, the head and tail folds fuse together at the level of the rudimentary right hind limb and therefore there is produced no trace of the posterior tube, which forms so conspicuous a feature of the chelonian amnion.

d). The epiblastic bridge of the connection seems to be persistent in Chelonia, while in the chick it is replaced by mesoblast and becomes extended on both sides.

e). There takes place no perforation of the sero-amniotic connection in Chelonia.

f). The sero-amniotic connection is quite without relation to the emergence of the young in the chick.

g). There is produced no albumen-sac in Chelonia and therefore the sero-amniotic connection is not enclosed within the foetal membranes. Consequently the allantois of Chelonia is very differently affected by the sero-amniotic connection.

II. Methods for the Preparation of Specimens.

From the second week of incubation it becomes almost hopeless to try to take out the contents of an egg entire in the fresh condition, and the whole egg-contents should therefore be hardened, in the following manner.

When an incubated egg is looked at by transmitted light it should, to be a promising one, be semi-opaque after the first week and entirely so after the first ten days. Having obtained such an egg it should be patiently tapped with a glass rod or something of the sort, until the greater part of the shell is broken into small pieces, the shell then taken off, piece by piece, leaving the underlying shell-membranes intact. From this point the procedure will be somewhat different according as the egg has been incubated more than two weeks or less.

a). When the egg has been in incubation less than two weeks. Care should be taken not to injure large blood vessels of the allantois, which is closely attached to the inner shell-membrane especially near the large pole: whenever they are injured by accident, the blood may be cooled by blowing at the point, when coagulation soon sets in. The shell is taken off in this way over an area about one inch square, and an opening made into the air chamber, the position of which is easily found out by the sound on tapping.

The egg is then put for a few minutes in Kleinenberg's picro-sulphuric acid in a cup of moderate size, to partly dissolve and soften the calcareous shell. The shell membranes then become easily separable from the allantois, and the whole contents can be entirely freed from the non-cellular envelopes. At this stage, the contents being yet soft, a large piece of the shell, disunited from the contents, should be left to support them in their relative positions until they have become hardened in Kleinenberg's fluid. Next day, the hardening fluid is replaced by alcohol, by taking it away in small portions by a pipette from the bottom of the cup, then gently pouring on alcohol (70 %) so that an alcoholic layer rests on it and repeating this process until it has been entirely replaced by alcohol. By this method Kleinenberg's fluid is easily and economically replaced by alcohol without disturbing the contents.

b). When the egg has been in incubation more than two weeks. There is here no danger of injuring the blood vessels, for the calcareous shell is now brittle and easily separable from the outer shell-membrane which is white, dry, and leathery. Besides, the contents are now firm and consequently the shell can be taken off very extensively, and even from the entire surface. However, the serous envelope being intimately attached to the inner shell-membrane, it is almost impossible to separate them in the fresh state without bleeding. But if we put the whole in Kleinenberg's fluid for ten to thirty minutes, there is no longer any danger. When, however, we wish to examine the cellular structure of the serous envelope, the inner shell-membrane should be left in position, for the safety of the underlying layers. When the contents are thus entirely cleared from the non-cellular envelopes, the specimen is put aside in Kleinenberg's fluid for about half a day. In this case, the contents, being firm the hardening fluid can be easily replaced by alcohol after it has done its work.

Finally, in both cases, (*a* and *b*) the vicinity of the connection should be freed from yolk, albumen, and the amniotic fluid, before the egg is many hours in alcohol, as otherwise these substances coagulate and can not then be removed. The outlines of the foetal membranes should, however, be examined or sketched before the last operation.

Explanations of Abbreviations and Colours used in Figures.

By the side of every figure, the age of the embryo, and the scale of enlargement are given. *Numerals in small type* on some of the surface views show the levels of the sections, represented in figures which are marked with the corresponding number. Abbreviations used in the plates are:—

Alb.	for	Albumen,
All.	,,	Allantois,
All. Vein	,,	Allantoic Vein,
Amn....	,,	Amnion,
Coel.′...	,,	Extra-Embryonic Coelomic Cavity,
Inall. Sep.	,,	Interallantoic Septum,
L. All. Art.	,,	Left Allantoic Artery,
Proam.	,,	Proamnion,
R. All. Art.	,,	Right Allantoic Artery,
S. A. C.	,,	Sero-Amniotic Connection,
Ser. Env....	,,	Serous Envelope,

Red represents in the surface views (Figs. 1-56) arteries—those of the inner allantoic limb being formed of dotted lines—and the purely mesoblastic part of the sero-amniotic connection. In the figures of sections (Figs. 57-94) *red* represents the protoplasma of mesoblastic cells:

Blue represents veins—those of the inner allantoic limb being in dotted lines. *Blue* also represents albumen in Figs. 51-56, and the hypoblast in the figures of sections (Figs. 57-94).

Black represents in the surface views (Figs. 1-56) the epiblastic bridge of the sero-amniotic connection and in the figures of sections (Figs. 57-94) the nuclei of cells.

Grey represents the interallantoic septa (Figs. 15-20). In the figures of sections it represents the epiblast.

Yellow represents yolk in Figs. 17-18.

Green represents the outer surface of the outer allantoic limb.

Plate XV.

(For explanation of abbreviations and colours see p. 370.)

FIGS. 1-5:—Dorsal views of embryos in different stages from 48 to 78 hours. The sero-amniotic connection in—

Fig. 1	is	represented	apart	in Fig.	38,
,, 2		,,	,,	,,	39,
,, 3		,,	,,	,,	40,
,, 4		,,	,,	,,	41,
,, 5		,,	,,	,,	42.

FIGS. 6-9:—Dorsal views of embryos in different stages from 104 to 168 hours. The embryo proper and the splanchnic mesoblast are left out. The serous envelope is also for the sake of simplicity not represented. The sero-amniotic connection in—

Fig. 7	is	represented	apart	in Fig.	43,
,, 8		,,	,,	,,	45,
,, 9		,,	,,	,,	46.

FIGS. 10-11:—The first is the surface view of an egg taken out from its non-cellular envelopes, and the second of the vicinity of the sero-amniotic connection in the same egg. The sero-amniotic connection has a peculiar fish-like shape. The anterior part of the sero-amniotic connection is in this specimen exceedingly elongated dorso-ventrally and being bent down on the plane of the paper appears triangular, looking like the tail fin of a fish. The connection is thus twisted at the point A. The sero-amniotic connection is represented apart in Fig. 47.

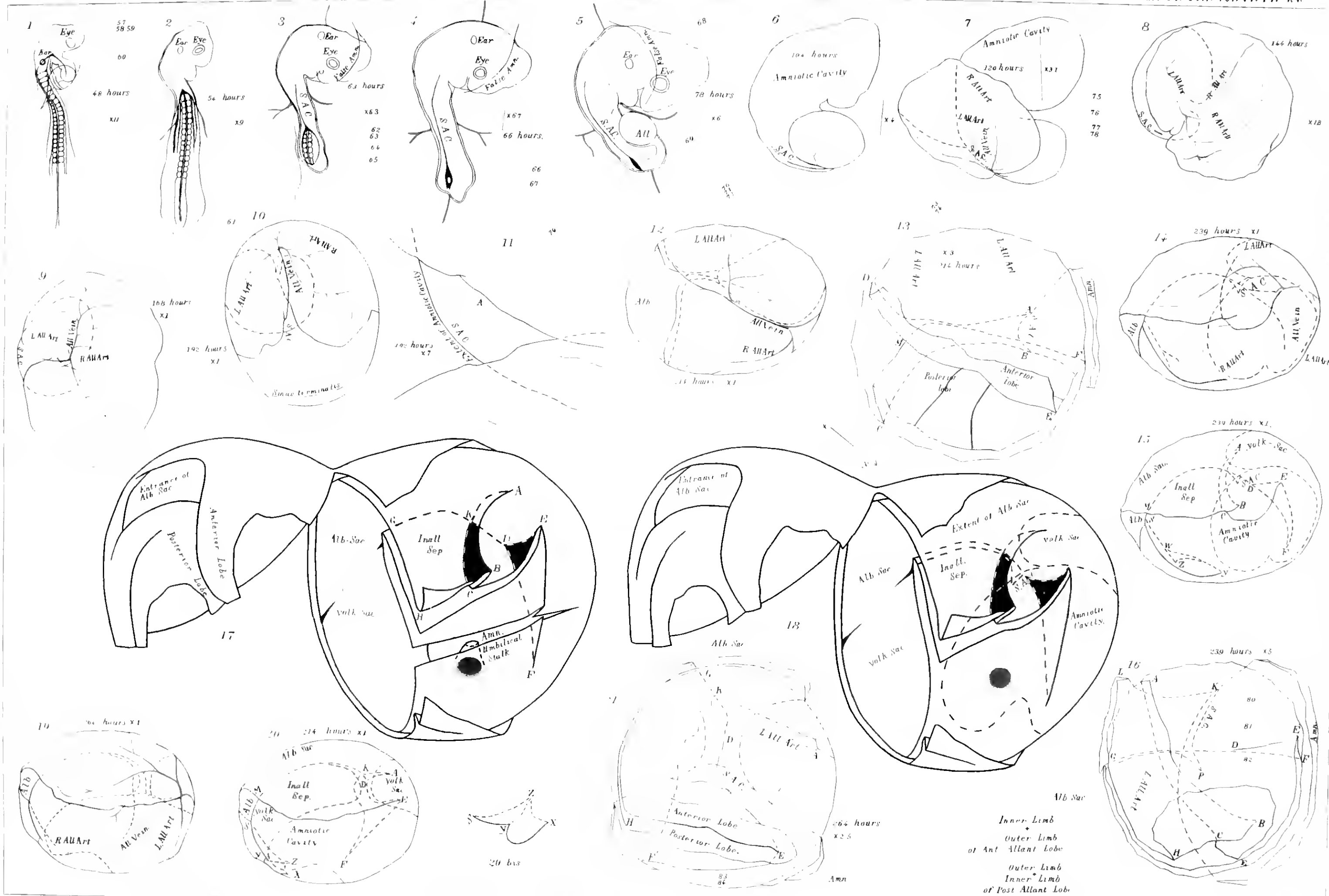
FIGS. 12-13:—The first is a surface view of an egg taken out from its non-cellular envelopes, and the second of the vicinity of the sero-amniotic connection of the same. The amniotic cavity, the cave-like diverticulum and the three allantoic septa are represented by dotted lines. In the second the area F″ E′ M F″ represents a part of the interallantoic septum and the space CBAD the cave-like diverticulum beneath the overlapping allantoic lobe. The sero-amniotic connection is represented isolated in Fig. 48.

FIGS. 14-16:—Fig. 14 is a surface view of an egg taken out from its non-cellular envelopes: Fig. 15 an outline of the foetal membranes; and Fig. 16 a surface

view of the vicinity of their sero-amniotic connection. The allantoic septa of Fig. 15 should be examined along with the blood vessels represented in Fig. 14 :—a full description of them is given in the text. The areas of the amniotic cavity and of the albumen-sac are represented by dotted lines, but a part of the yolk-sac is concealed beneath the albumen-sac. In Fig. 16, the albumen-sac is advanced as a cave-like diverticulum, GPKL, towards the amnion, which it meets along the line KP, the sero-amniotic connection. The area DE″E′CHGDF″E″ is a part of the interallantoic septum, which is folded along the line DE″. The sero-amniotic connection is represented isolated in Fig. 49.

FIGS. 17-18 :—The first is a diagrammatic representation of the allantoic septa alone, and the second of all the foetal membranes of the egg represented in Figs. 14 and 15. In both, the outer allantoic limb is removed to a great extent. In Fig. 17 the folded area FEDGHCE is a part of the interallantoic septum. The line ED is the axis along which the interallantoic septum is folded in two. The area ABC is the minor septum formed by the left allantoic artery. The septum formed by the right allantoic artery is not represented. The deep black areas represent duplicated parts of the septa. The positions of the albumen-sac, the yolk-sac, the umbilical stalk, and the amniotic cavity are shown by artificial incisions. In Fig. 18 the areas of the amniotic cavity and albumen-sac are given in addition, but a part of the yolk-sac is concealed beneath the cave-like diverticulum of the albumen-sac.

FIGS. 19-21 :—Fig. 19 is a surface view of an egg, 264 hours old, taken out from its non-cellular envelopes. Fig. 20 represents the outlines of the foetal membranes; Fig. 20 *bis* the minor septa formed by the right allantoic artery as seen from the surface; and Fig. 21 the surface view of the vicinity of the sero-amniotic connection. In examining the allantoic septa of Fig. 20 reference should be made to the blood vessels represented in Fig. 19. The areas of the amniotic cavity and albumen-sac are represented by dotted lines, but a part of the yolk-sac is concealed beneath the cave-like diverticulum of the albumen-sac. In Fig. 21 the areas KDEF′K and KGHE are part of the folded interallantoic septum, and the areas ADK and XZSY (Fig. 20) are the minor septa formed by the left and right allantoic artery respectively. The sero-amniotic connection is represented isolated in Fig. 50.



S. Hirota del.

Lith.& Imp. Koshiba.

Plate XVI.

(For explanation of abbreviations and colours see p. 370)

FIGS. 22-36:—These are surface views of three eggs aged respectively 312, 385, and 432 hours. They are arranged mostly in three horizontal rows, each row being devoted to one egg. The first two coloured figures of a row are the surface views of opposite faces of the same egg, taken out from its non-cellular envelopes, and the two last figures of a row are the outlines of the foetal membranes of the corresponding faces. In following a blood vessel or any other structure from one face to the other in these rows, it should be remembered that the lower edge of the first and third figures of a row should be applied to the upper edge of the second and fourth figures. It should also be noticed that as an egg has a considerable thickness, a structure leaving one face may not be found entering at exactly the corresponding point of the other. Each of Figs. 26, 31, and 36 represents the allantoic septa, come together at the vegetable pole of the respective egg, as seen from that pole. Full lines are tracings of the septa on the outer limb of the allantois. In Figs. 27-28 and Figs. 32-33, the branches of the allantoic vein are anastomosed. In the example given in Figs. 32-36 the interallantoic septum is not folded on itself. In these examples of later stages the interallantoic septum is not influenced by the left allantoic artery. The sero-amniotic connection of the specimen represented in —

Fig.	22-26	is represented	apart	in Fig.	51.
,,	27-31	,,	,,	,,	54.
,,	32-36	,,	,,	,,	56.

FIG. 37:—Dorsal view of the vicinity of the sero-amniotic connection of an embryo in the same stage as that represented in Fig. 7. The dotted lines show the extension of the lateral fold formed within the amniotic cavity (see Diag. 5). The points A, B, & C correspond to those marked with the same letters in the section (Fig. 92) of that level. The blood vessels in the inner allantoic limb are for the sake of simplicity not represented.

FIGS. 38-42:—Dorsal views of the sero-amniotic connection represented in Figs. 1, 2, 3, 4, and 5 respectively. The partly obliterated mesoblastic septum is represented by a broken line. All are magnified to the same scale (× 10).

FIGS. 43-50 :—The mesoblastic sero-amniotic connection at the corresponding stages projected on an imaginary straight line, showing its relative width in different individuals and in different parts of the same individual. In breadth all are stretched as much as possible, but in length, each represents the shortest distance between the two extremities of the connection. The ends, where the connection is incomplete, are represented by transverse interruptions. The black lines in the middle represent the epiblastic sero-amniotic connection, their interruptions showing breaks in it. All are magnified to the same extent (× 10). The connection represented in—

Fig. 43 is that of Fig. 7.
,, 44 is from a specimen, 144 hours old.
,, 45 is that of Fig. 8.
,, 46 ,, ,, 9.
,, 47 ,, ,, 10,
,, 48 ,, ,, 11,
,, 49 ,, ,, 14,
,, 50 ,, ,, 19.

FIGS. 51-56 :—Representations of the perforated states of the sero-amniotic connection, as seen from the amniotic cavity. The albumen-sac approaches the amnion as a sort of tube or cave. Albumen is coloured blue. The line AA is the line along which the interallantoic septum is continuous with the inner allantoic limb. The circular line (Amn.) represents a part of the amnion. Differently magnified. The specimen represented in—

Fig. 51 is that represented in Figs. 22-26.
,, 52 is from an egg fourteen days old.
,, 53 ,, ,, fifteen ,, ,, .
,, 54 is that represented in Figs. 27-31.
,, 55 is from an egg seventeen days old.
,, 56 is that represented in Figs, 32-36.


L. All. Art
Inall. Sep
Yolk Sac
Inall. Sep
Inall. Sep

PLATE XVII.

Plate XVII.

(For explanation of abbreviations and colours see p. 370)

All the figures are transverse sections of the embryos cut through the levels marked with numerals in small type in the corresponding surface views.

FIGS. 57-60 are taken from the specimen represented in Fig. 1. The mesoblastic septum in Fig. 58 is magnified in Fig. 59.

FIG. 61 is taken from the specimen represented in Fig. 2.

FIGS. 62-65 are ,, ,, ,, ,, 3.

,, 66-67 ,, ,, ,, ,, ,, 4.

,, 68-69 ,, ,, ,, ,, ,, 5,

,, 70-72 ,, ,, ,, ,, ,, 6.

FIG. 71. An epiblastic cell in the connection is here in division.

FIGS. 73-74 are taken from the specimen represented in Fig. 7.

,, 75-78 ,, ,, ,, ,, ,, ,, 8.

FIG. 78. An epiblastic cell is here in division.

FIG. 79 is taken from the specimen represented in Fig. 11.

The dorso-ventrally elongated sero-amniotic connection is pressed down to a horizontal position.

FIGS. 80-82 are taken from the specimen represented in Fig. 16,

,, 83-84 ,, ,, ,, ,, ,, ,, 21.

The connection of Fig. 83 is magnified in Fig. 84.

FIGS. 85-87 are taken from the specimen of the twelfth day showing perforations in the sero-amniotic connection.

FIGS. 88-89: Represent a stage nearly equal to that shown in Fig. 5. The first is anterior to the second by $\frac{1}{10}$ *mm*. At the level given in Fig. 89 the amnion is just closed. In these two figures nuclei are represented by white spots.

FIG. 90 Section just in front of the allantois of an embryo, which is of the same stage as that represented in Fig. 5. The point A is the lateral angle of the amniotic fold.

FIGS. 92-93: Sections of the specimen represented in Fig. 37, through the levels marked with the corresponding number on the latter. The points A, B, & C inFig. 92 correspond to the points marked with the same letters in the surface view.

On a New Human Tape-worm (Bothriocephalus sp.).

by

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and

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With Plate XVIII.

The human tape-worm to be noticed in this paper is, as will be borne out in the sequel, a very large *Bothriocephalus* species furnished with double genital ducts and openings in each proglottis. So far as we are aware, any similarly characterized species of that genus has not hitherto been included in the human parasitic fauna, though a few such have long been known to occur in seals and certain fishes. We hold it highly probable that this human *Bothriocephalus* is a near relative of those already described from seals,[1] and possibly identical with one of them. The supposition that the patient who discharged it, had acquired his from a source similar to that which furnishes seals with theirs, seems to be not improbable from the fact of his continual residence near the sea-shore.

The specific determination of this tape-worm, whether *species novarum* or not, we have preferred to leave to the discrimination of

1 *Bothr. variabilis* Krabbe, *Bothr. fasciatus* Krabbe and *Bothr. tetrapterus* v. Siebold, from several species of *Phoca*.

those investigators more favourably situated than we are with respect to literature and specimens to compare with.

We owe our material to the courtesy of Mr. Sōichiro Nakamura, who, in 1892, while acting as physician to the hospital attached to the Takashima Coal Mines, near Nagasaki, obtained the worm from one of his patients.

Particulars concerning this patient, partly furnished us by Mr. Nakamura and partly obtained by one of us from the patient's family, run to the following effect:

Tamaji Murazato, male, born 1865, at Taira-mura (a village on the Ariake-Sea, near the town of Shimabara) in the Province Hizen. In boyhood healthy but never muscular. Remained in his native village until 1879, when he went to Nagasaki. Here attacked by cholera but recovered. Up to 1892 resided at several places in the neighbourhood of Nagasaki and other sea-side localities within the Province Hizen, outside of which he seems to have never travelled. Calling: emanuensis, post-office clerk, school-teacher, &c. 1891 settled at the Takashima Coal Mines, where he had been engaged in book-keeping business until his death by accident in November of the following year. Some five years previous to this period, he began to suffer occasional dizziness and colic. Medical help had not much effect, beyond palliating the latter. Gradual anæmia supervened. During October 1892, a piece of tape-worm about one foot long was discharged. About this time violent colic is said to have returned. He was then taken into the hospital before mentioned and submitted to medical treatment by Mr. Nakamura, whose notes taken at the time are as follows:

"Patient aged 28 years. Medium bodily constitution. Badly nourished, weary. Symptoms of cyanosis on face. Liable to fall into insensibility while sitting or otherwise occupied. Pulse weak

and frequent, numbering 120. Palpitation somewhat accelerated. Temperature 36.8° C. Tongue with yellowish covering. Appetite ordinary, sometimes vigorous. Gastric region swollen out and frequently giving spasmodic pain, radiating towards the back and ceasing gradually or suddenly, followed by a feeling of pressure on intestines. This feeling either remains at one place or shifts its position. The attack occurs after taking food but also at other times. Pressing the gastric region from outside has soothing effect on the pain. Sometimes pain also in the pelvic region. Diarrhœa, or costiveness for many days.

"From above symptoms, the presence of *Ancylostomum duodenale* was suspected. Microscopical examination of the fæces however unexpectedly revealed a number of eggs, resembling very much those of *Distomum ringeri* both in size and appearance. Irrespective of what parasite these eggs might belong to, a dose of *extr. filic. mas.* was tried and the result was the discharge of a tape-worm measuring *10 meters* in length and, at the broadest portion, 25 millimeters in breadth. The broad hind end had its extreme tip shrunk, much macerated and easily detachable. Of the other end, a portion as thin as 1.5 millimeter was found but no head could be discovered. From the following day, all the complaints the man had suffered from for so many years entirely disappeared."

But the man did not live long to enjoy this relief, for during the following month a collision with a coal truck broke his back and killed him. Post mortem examination was not allowed to Mr. Nakamura notwithstanding his appeal.

Judging from Mr. Nakamura's statements, the tape-worm, a *Bothriocephalus* as already mentioned, must have been an extraordinarily long and broad one, gradually tapering anteriorly into an almost filamentous collum.

We did not see the entire specimen, but sample pieces from four different regions of the body were kindly placed at our disposal. They were preserved in strong spirit that hardened them into stiffness but kept the tissues in excellent condition.

Sample No. 1 is a piece from near the anterior end of the original specimen. It measures 3 mm. in breadth and 0.5 mm. in thickness at the middle. No trace of reproductive organs is visible in these segments.

Sample No. 2 is a piece somewhere from the anterior quarter of the original specimen. At this region the body already presents considerable dimensions, being 13 mm. broad and about 1.5 mm. thick. The reproductive organs are partly developed; but of this, later on.

Sample No. 3 is from the middle portion, 14–16 mm. broad and about 1.5 mm. thick. The reproductive organs are fully developed and the uterus is already partially filled with eggs.

Sample No. 4 consists of two pieces from the posterior portion, one of them cut off 40 cm. from the hind end. Breadth varies from 10 mm. to 15 mm; thickness measures 1.5 mm. or somewhat more. The varying breadth is certainly due to different states of contraction and accordingly, where the breadth is less, the proglottis is comparatively longer. The uteri in this section are much distended and filled up with eggs.

Mr. Nakamura's measurement of maximum breadth, namely 25 mm., was no doubt taken whem the worm was quite fresh. This accounts for the fact that nowhere in the alcoholised and contracted samples before us is that great breadth attained. The foremost portion of the original specimen, stated by Nakamura to have been only 1.5 mm. broad, must have belonged to a section more anteriorly situated than our sample No. 1.

One of the very striking features of our *Bothriocephalus* species

is the extreme shortness and consequent narrowness in antero-posterior direction, of the proglottides. They almost present the appearance of closely set transverse wrinkles to the naked eye (*vide* figs. 1 and 2, Pl. XVIII). In the middle and hind regions, the length of the proglottides averaged only 0.45 mm., and their breadth 14–16 mm. Even where most distended (breadth 10 mm.), their length did not exceed 0.8 mm. The average length given above was calculated by counting *from the outside* the number of proglottides within a measured space *along the median line*. It is important to note here that what appeared externally to be two distinct and consecutive proglottides as indicated by the usual boundaries, very often proved to be one internally, i. e. with respect to certain genital arrangements. For instance, a piece 30 mm. long from sample No. 4, showed but 57 uteri in a series, while the number of proglottides as counted in the way mentioned amounted to about 68. Besides the superficial supernumerary boundaries, both dorsal and ventral, that extend throughout the entire breadth of the body, others which disappear at the middle of it after running for a greater or less distance from the margin, are of quite frequent occurrence; so that, counting the proglottides by the marginal serration would give a still greater number than when counted along the median region. Thus, in the piece above referred to, the number of segments as counted near the margin amounted to 93 or thereabout. The proglottis that has such an incomplete supernumerary boundary on the one side generally shows the same also on the other. Sometimes such a false proglottidal boundary in its course joins a neighbouring true one or loses itself on the general surface to appear again after a short interruption. In rare instances, apparently two incomplete false boundaries in succession were interposed between two true ones in the marginal region. The antero-posterior lengths of consecutive segments, separated by false proglottidal boundaries

and belonging to one internal proglottis, are quite variable but when taken together may be said as approximately equalling or somewhat surpassing the length of those proglottides that show no trace of false demarcations whatever (Figs. 4, 5, and 6). The features of marginal serration as also the manner of indentation of the body-surface essentially agree in all proglottidal boundaries, both true and false, excluding all possibility that the latter might be some mechanical or accidental production. We are inclined to view the phenomenon in the light that the present species of Cestodes has a tendency to produce superficially more proglottides than it does internally, contrary to the well-known case of *Ligulidae* in which the external strobilation remains more or less obsolete. In other words, under the crowded state of proglottides in our *Bothriocephalus* species, one proglottis seems to remain but partially developed, i. e. only superficially marked, in order to give necessary space for the full development of certain genital parts (especially uterus, cirrus, and ovary) in its immediate neighbour. The widely distributed testes and yolk-glands develope themselves as well in the abortive as in the other proglottis and are apparently related to the genital ducts and openings of the latter as if they were its own.

What further seemed to us to be of interest with respect to the strobilation of the present species, is the presence, in our sample No. 1, of indications that certain proglottides are undergoing repeated division. In this anterior region as many as 38 proglottides were counted within a space of 10 mm., giving to each proglottis an average length of 0.26 mm. (by 3 mm. in breadth). The actual lengths were tolerably uniform, except only that the latest formed proglottides were only half or less than half as long as the others. Division of a proglottis into two takes place, not at its middle, but invariably at its anterior portion; consequently, of the two

new proglottides, the anteriorly situated is always the shorter until the normal length is attained by growth. Examining with a hand-lens of low magnifying power, our attention was at once called to the facts that the boundaries of proglottides were not all alike in their sharpness and depths of marginal indentation and that they succeeded one another with a certain degree of regularity. The differences are evidently due to the oldness or lateness of their formation. Usually four or five consecutive proglottides together formed a group or what might conveniently be called a primary segment, terminated anteriorly and posteriorly by much better defined proglottidal boundaries. In other words, every fourth or fifth boundary was generally the most pronounced and presumably the oldest formed. Where four proglottides made up such a primary segment, the boundary between its 2nd and 3rd proglottides was usually the next well-defined, whereas that between the 1st and 2nd and also that between the 3rd and 4th were comparatively somewhat less sharply pronounced. We might interpret this so, that such a primary segment is composed of two secondary segments, each of which again consists of two tertiary segments or proglottides. Often the front proglottis of a primary segment had more or less distinctly divided into two, at apparently a quite late period, in which case that primary segment seemed to consist of five, instead of four proglottides. Sometimes the next tertiary proglottis also showed signs of a similar quaternary division. Thus then, within a given primary segment, the division of proglottides takes place successively backward beginning from the foremost proglottis. This corresponds to the fact already mentioned that in an individual proglottis division occurs at its anterier portion. It is easy to conceive that by a continued process of such division, an individual proglottis would in course of time come to rank as a secondary and this again as a primary

segment. But this process seems not to take place uniformly throughout, as indicated by the fact that primary segments, with and without proglottides undergoing quaternary division, showed no regularity in their order of succession, and also by the fact that sometimes, between two more or less typical primary segments, there were interposed two or more proglottides, which showed no trace of division and were separated from one another by boundaries as marked, and hence presumably as old, as those that bounded any primary segment. At all events, the generally accepted idea that in a tape-worm the more posteriorly situated proglottis is always the older, seems not to hold true in the present species of *Bothriocephalus*. The repeated serial division here described naturally reminds one of a somewhat similar process in *Microstomum* and certain annelids, but we abstain here from entering into comparisons. We regret that Fig. 11, Plate XVIII, which should represent a portion of sample No. 1, has failed to illustrate exactly the ordinal distinctions of proglottidal boundaries plainly visible on the real object.

How far backwards in the entire tape-worm the subdivision of proglottides is repeated, could not be ascertained further than that it probably ceases with the begiuning of the development of genital ducts, somewhere between the two portions represented by our samples No. 1 and No. 2. In the latter, in which the formation of genital ducts is almost completed, abortive proglottides before described are plentifully met with. One might regard these as indications of divisions taking place here, though certainly only external, were it not for the fact that similar abortive proglottides are also found in sample No. 3 or No. 4 in about the same proportion. Hence, we rather consider all those present in our samples No. 2–4, as representing the proglottides that were formed about the time when the genital ducts were beginning to develope but too late to develope their own.

Moreover, the body of the present *Bothriocephalus* species is longitudinally traversed by several, more or less deep, furrows on both its ventral and dorsal surfaces. These are few and insignificant on sample No. 1, but numerous on all other samples, in which the most conspicuous are the two on either surface, that run almost uninterruptedly and parallel to each other along the double series of main genital ducts, dividing the tape-worm body into a middle and two lateral longitudinal zones (see figs.). They are slightly nearer to each other than to either body-margin. They may attain the depth of about $^1/_4$ the thickness of the body and must plainly be constantly present in fresh specimens. The same can hardly be asserted of all other longitudinal furrows seen on the middle and lateral zones above mentioned, which are, as seen in alcoholic samples, of quite variable depths and sharpness, often interrupted or losing themselves in their course, and by no means definite in their number. However, some 3–5 in the middle and some 5–7 in the lateral zone are the usual numbers to be met with.

As already indicated, there are, to each (true) proglottis, two sets of genital openings, situated right and left and communicating externally at the bottom of the two most conspicuous longitudinal furrows of the ventral surface (*b*, *b*, figs. 5 and 6). Each set consists, in antero-posterior succession, of a cirrus (*cir. o.*, figs. 10 and 12), a vaginal (*vag. o.*) and an uterine (*ut. o.*) opening, lying close to one another. On account of their secluded position within the longitudinal furrow, they are usually not recognizable from the outside, but a pit-like depression of the latter, associated with a short cross-furrow or two, sufficiently marks their position and at the same time serves as the index to distinguish the ventral from the dorsal surface. In many proglottides of sample No. 4, the cirrus is externally visible as a minute rounded protrusion, evidently the result of its partial evagination (fig. 6).

As might be inferred from the above mention of genital openings, the arrangements of sexual organs in the present species are typically bothriocephaline. To begin with the male organs:

These develope earlier than the female sexual organs, as in other species of Cestodes. The testes, which present the usual features, are present from sample No. 2 downwards. In the sample just mentioned they are not yet fully mature and are separated from one another by somewhat wider spaces than in sample No. 3 or 4, in either of which the production of spermatozoa is actively going on. They may attain a diameter of 0.07 mm. Generally arranged in a single layer, they occupy the usual position in the "*Mittelschicht*" (*h*, fig. 8). The area of their horizontal distribution is divided into three parts by the regions taken up by the double sets of main genital ducts. In cross-sections passing midway between the anterior and posterior limits of proglottides, we have counted 30–40 testicular vesicles in each of the three parts.

The cirrus (*cir.*) is a round or oval-shaped body, essentially agreeing in its fine structure with the same organ of other *Bothriocephalus* species. It lies with the axis of its tortuous lumen slightly inclined from above downwards in an antero-posterior direction (fig. 12). At its superior end, the cirrus is directly continuous with the muscular wall of a spherical *vesicula seminalis*. The latter presents an appearance as though it were a posteriorly bent, knob-like, terminal portion of the cirrus itself. Its cavity, as also the lumen of the cirrus, is narrow and empty in sample No. 2, but filled up with and much distended by spermatozoa in samples No. 3 and No. 4. In these the seminal vesicle measures about 0.12 mm. and the approximately spherical cirrus about 0.25 mm. in diameter. In sample No. 2 both are much smaller, the cirrus being here decidedly oval-shaped as seen in fig. 12. (In this figure, the lettering *cir.* referring to the second

cirrus from left, points to the *vesicula seminalis*, which is better seen on the cirrus next to the right.)

The *vas deferens* (*vd.*, fig. 10) is seen in sample No. 2 as a thin cellular string with no recognizable lumen and in samples No. 3 and No. 4 as a thin-walled, much convoluted tube filled with spermatozoa. After starting from the *vesicula seminalis* it turns towards the median line, i. e., to the right if it belongs to the left set of genital ducts, and *vice versa*. It then pursues its irregular course for a variable distance but always stopping short before the end of the first uterine loop of the corresponding side is reached. How it communicates with the testes could not be observed.—

Of the female sexual organs, the *vitellarium* (*dts.*, fig. 8) is yet very scantily and weakly developed in sample No. 2, but fully developed in samples No. 3 and No. 4. In these it consists of very numerous lobules about 0.025 mm. broad and about 0.045 mm. long, with their long axis directed perpendicularly to the body-surface. Arranged in a single layer, they occupy the usual position outside of the layer of strongly developed longitudinal muscles and are much more numerous on the ventral than on the dorsal side. On the former they are distributed almost uniformly throughout except at regions occupied by the two sets of main genital ducts (fig. 7). At the body-margin they pass in a continuous layer into those of the dorsal side, where, tracing them in the median direction, they become scarcer and irregularly distributed, and finally disappear at a greater or less distance from the position of the uteri. In the middle region of the dorsal side, many proglottides showed none of them, while in others they were developed also in this part but in irregular patches comparatively few in number.

The ovary is present to each set of genital ducts. It is a horizontally and transversely stretched cellular string that divides on

either side into an elongated bundle of anastomosing branches (*ov.*, fig. 10). At its middle, whence the oviduct arises, it measures only 0.01 mm. or less in thickness while the lateral, divided portion may be nearly 0.08 mm. broad. In functional activity, it may measure 1 mm. from the origin of the oviduct to either of its extremities. Its position in the proglottis is on the ventral side of the *Mittelschicht*, along the posterior proglottidal border.

The exact course of the female ducts could only be conveniently studied by combining sections of sample No. 2, where they were yet without contents except some masses of yolk-like granules that occurred here and there in the uterus. The oviduct (*ovd.*, fig. 10), after its origin from the ovary, at first runs backwards, soon to take an irregular ascending course, which is at the same time directed outwards, i. e., away from the median line of the tape-worm. During this course, the oviduct is joined first by the vagina that approaches it from the median side and then, after a short interval, by the yolk-duct that comes up from below. In tracing the lumen of the oviduct from the ovary, it appears to be directly continuous into that of the vagina (as shown by the unshaded passage in fig. 10), rather than into the remaining portion of the oviduct leading towards the junction of the yolk-duct.

The vagina (*vag.*) is a fine duct, that makes a few windings beneath the uterus but crosses the ovary on its dorsal side. Posteriorly and close to the junction with the oviduct, the vaginal tube swells up into an oblong vesicle, which we consider to be the *receptaculum seminis* (*sb.*, fig. 10). Anteriorly it descends almost perpendicularly along the posterior border of the cirrus and opens at the vaginal pore (*vag. o.*, fig. 12) just behind the cirrus opening.

The yolk-duct (*dtg.*, fig. 10) is a very thin tube that after descending a short distance from the oviduct soon becomes untraceable.

Although yolk-granules were often found in its lumen, we could never make out its connection with the vitellarium. Nor could we ever recognize the shell-glands. Nevertheless there can be no doubt that the present species essentially agrees in these as in so many other points with *Bothriocephalus latus*.

The uterus (*ut.*, figs. 10 and 12), as seen in sample No. 2, occupies for the most part the dorsal region of the *Mittelschicht*. Its anterior portion descends ventrally, almost in a straight line to open at the uterine pore situated at about the middle of the length of the proglottis (*ut. o.*, fig. 12). The rest of the uterus, notwithstanding some irregularity in its course, describes in general two loops on each side. The distance between the ends of two opposite loops measures only about half a millimeter. No eggs are yet found and the uterine lumen is for the greater part obsolete.

In sample No. 3, the production of eggs is considerably advanced. They fill the uterus and distend the latter into a wide tube, but not by far to such a great extent as in sample No. 4. The latter, when viewed by holding it against the light, shows the uteri of successive proglottides as blackish spots, about 1 mm. or slightly more in breadth and arranged in two longitudinal series. In each series some may lie somewhat to the right or left of the general line. When compressed and clarified, they are certainly very distinctly visible. In such preparations they present variable forms but the most general and the least disturbed condition is that figured in fig. 7. Each uterus appears as consisting of two pairs of lobes, corresponding to the four loops of the uterine tube in sample No. 2. The anterior pair is the shorter of the two and clasps the posterior side of the cirrus. The numerous eggs contained in the uterus, or more properly their shells, give colour to the uterine lobes which varies from the colourless transparency of the portion nearest the oviduct into the dark brown of

the portion containing older eggs.

An egg, with brown shell and taken from the uterus near its external opening, is represented in fig. 9. In alcohol the shell is collapsed but soon returns to its proper form by leaving it a while in water. It is rather thick. The general form is oval, 0.063 mm. long and 0.048–0.05 mm. broad. The diameter of the operculum measures about 0.02 mm. The contents are oil-globules and a morula-like mass of finely granular spheres, evidently cleavage-spheres.—

With regard to other points in the structure of the present *Bothriocephalus* species, we have made but casual observations. The main longitudinal nerve runs outside the series of uteri, on each side of the body (*ln.*, fig. 8). It lies nearer to the uteri than to the body-margin of the same side. A short distance inside of the longitudinal nerve, the main trunk of the excretory vessels finds its course on both sides. Sections of excretory vessels were sometimes also met with at other places. The muscular system is essentially the same as in *Bothriocephalus latus*. Finally, we ought to mention that nowhere in our samples have we found a single calcareous body in the mesenchyma.

Explanation of Plate XVIII.

a, dorsal groove along the series of main genital ducts.
b, ventral groove along the same.
cir, cirrus.
cir. o., cirrus opening.
dtg., yolk-duct.
dts., yolk-gland or vitellarium.
h., testes.
lm., longitudinal muscular layer.
ln., longitudinal nerve.
m—m., median-line of the specimen.
or., ovary.
ord., oviduct.
sb., receptaculum seminis.
tm., transverse muscular layer.
ut., uterus.
ut. o., uterine opening.
vag., vagina.
vag. o., vaginal opening.
vd., vas deferens.

Fig. 1. A piece from sample No. 4, 40 cm. from the hind end of the original specimen. Dorsal view. Nat. size.

Fig. 2. Ventral view of the same. Nat. size.

Fig. 3. Cross-section of the same. Nat. size. The two black spots represent the uteri filled with eggs.

Fig. 4. Dorsal view of a portion of the same. Magnified 10 times.

Fig. 5. Ventral view of a portion of the same. Magnified 10 times.

Fig. 6. Ventral view of another portion of the same, with cirri partially protruded. Magn. 10 times. By an oversight this figure has been placed in a position the reverse of that intended.

Fig. 7. Portion of a preparation from sample No. 4, seen from the ventral side. Magn. 20 times. The piece was somewhat compressed between two glass-plates, coloured and clarified. The numerous dot-like bodies on either side of the uteri represent yolk-gland lobules of the ventral side.

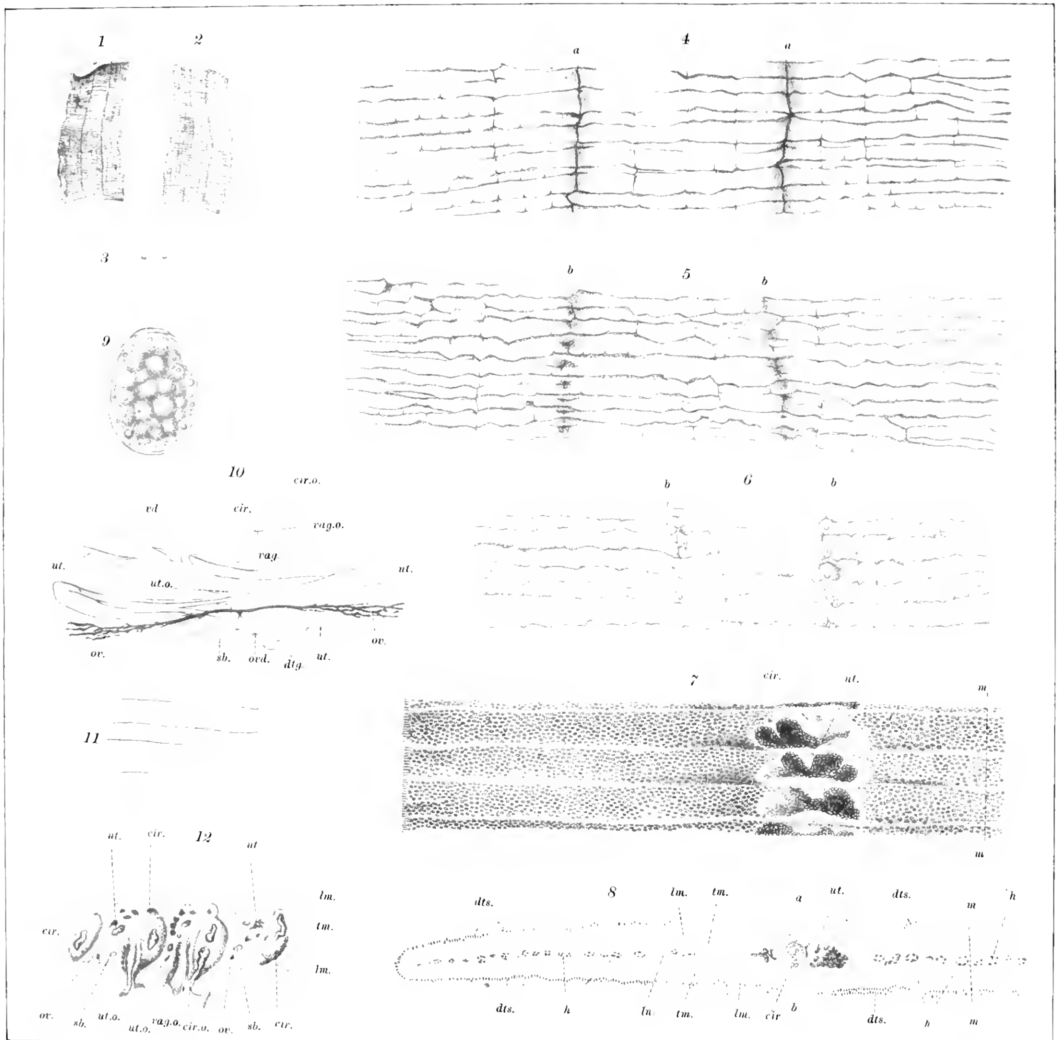
Fig. 8. Portion of a cross-section of sample No. 4. Magn. 20 times.

Fig. 9. An egg taken from the uterus. Magn. 440 times.

Fig. 10. Half-diagrammatic representation of a left set of main genital ducts, as seen from the ventral side; made out by combining sections of sample No. 2. Magn. 150 times.

Fig. 11. A portion of sample No. 1. Magn. 10 times. This figure is a failure in so far as it does not properly show the ordinal distinctions of proglottidal boundaries.

Fig. 12. Portion of longitudinal section passing through the cirri, from sample No. 2.



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